GEOMETRY OF A SYSTEM OF LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

Using a system of partial differential equations, in this paper we shall study a configuration of surfaces and congruences of lines whose elements are in a 1 — 1 correspondence. Many eminent geometers such as Wilczynski, Lane and Green had investigated the properties of a single congruence of lines but very few have studied a pair of congruences.

In this work we shall consider a pair of congruences whose generators are in 1 — 1 correspondence and which are related in a special way. We shall make use of the properties of integrable systems of differential equations and the theory of transformation to explore the intrinsic nature of the new configuration.

The Elements of the Theory of Surfaces

The concept of surface is basic in this study. Roughly speaking, a surface is a two-parameter family of points; that is, it is the locus of a point moving with two degrees of freedom. For the purposes of this study, however, these descriptions are not altogether adequate. We shall need more precise definitions.

Let the homogeneous projective coordinates (x_1, x_2, x_3, x_4) of a point in a three dimensional space be given as single-valued analytic functions of two independent variables u and v by

$$x_i = x_i(u, v), i = 1, 2, 3, 4,$$

which we write in vector notation as:

$$x = x(u, v).$$

Curves on a Surface

As u and v vary in a domain T the point generates a surface — an analytic surface. The point shall be denoted by P_x and its locus by S_x.

A curve is a one parameter family of points. If we put v = constant, c, in the parametric vector equation $x = x(u,v)$ of S_x we get a one-parameter set of points whose vector equation is

$$x = x(u, c) = f(u).$$

The locus of the point P_x is a curve which we shall call a u -curve on the surface S_x. If c varies over all possible values we obtain a family of u-curves which covers S_x.

A second family of curves is obtained if we set u = constant = c. It is the locus of P_x where

$$x = x(c, v) = g(v).$$

If c takes all possible values a second family of curves — the v-curves — is obtained.

The surface S_x is then covered by two families of curves — the u-curves and the v-curves. Thru each point P_x intersect one u-curve and one v-curve. The parameters u and v are called the curvillinear coordinates of P_x.

Let C be any curve on the surface S_x. Then it is a one parameter family of points. Therefore, each of the coordinates u and v of a point on c is expressible as a function of a parameter t; that is,

$$u = u(t), v = v(t).$$

Thus C is the locus of points P_x, where

$$x = x(u(t), v(t)),$$

as t varies over all possible values.

Consider a point P_x on C with coordinates u(t), v(t). The line $P_xP_{x'}$, where $x = x(u(t), v(t))$ and $x' = \frac{\partial x}{\partial u} \frac{du}{dt} + \frac{\partial x}{\partial v} \frac{dv}{dt}$ is tangent to C at point P_x.

In the case of the u-curve and the v-curve the lines $P_xP_{x_u}$ and $P_xP_{x_v}$ are tangent lines to these curves respectively at P_x.

A differential equation of the form

$$M(u, v)du + N(u, v)\,dv = 0$$

represents a one-parameter family of curves on the surface. For, if

$$v = f(u, c), \text{ c is a variable parameter,}$$

is a general solution of the differential equation, from the vector equation $x = x(u, v)$ of S_x, we get, when v is replaced by f(u, c), the vector equation

$$x = x(u, f(c)) = g(u,c),$$

which is the vector equation of a one-parameter family of curves on S_x.

In particular, the parametric curves u = constant and v = constant are integrals of du = 0 and dv = 0 or of dudv = 0.

Tangent Plane and Osculating Plane

The plane determined by the points P_x, P_{x_u} and P_{x_v} is called the *tangent plane of S_x at P_x*.

If C is a curve on S_x thru P_x defined by

$$u = u(t), \; v = v(t),$$

then the plane determined by P_x, $P_x{}'$ and $P_x{}''$, where

$$x' = x_u \frac{du}{dt} + x_v \frac{dv}{dt}$$

$$x'' = x_{uu} \frac{dy^2}{dt} + 2x_{uv} \frac{du}{dt} \frac{dv}{dt} + x_{vv} \frac{dv^2}{dt} + x_u \frac{d^2u}{dt^2} + x_v \frac{d^2v}{dt^2},$$

is called the *osculating plane of C at P_x*.

If at each point P_x of C on S_x, the tangent plane of S_x and the osculating plane of C *coincide*, then C is an *asymptotic curve* on S_x.

If $\underline{\bar{X}}$ is any point in the tangent plane of S_x or in the osculating plane of C at P_x then

$$\begin{aligned} (\underline{\bar{X}}, x, x_u, x_v) &= 0 \\ (\underline{X}, x, x', x'') &= 0 \end{aligned} \qquad (2.2)$$

are the equations of the tangent plane of S_x and the osculating plane of C respectively at P_x.

We say that a curve C on a surface S_x is an *asymptotic curve* if the tangent plane of S_x at each point of C coincides with the osculating plane of C.

Since P_x and $P_x{}'$ are points in both the planes in (2.2), a necessary and sufficient condition that C is an asymptotic curve is that the point $P_x{}''$ also lies in the tangent plane.

This means that

$$(x'' \; x \; x_u \; x_v) = 0.$$

When the value of x'' given by (2.1) is substituted in this equation we get

$$L\,du^2 + 2M\ du\ dv + N\ dv^2 = 0, \qquad (2.3$$

where

$$L = (x_{uv},\ x,\ x_u,\ x_v) \qquad (2.4$$

$$M = (x_{uv},\ x,\ x_u,\ x_v)$$

$$N = (x_{uv},\ x,\ x_u,\ x_v).$$

The differential equation of the parametric net on S_x

$$du\ dv = 0. \qquad (2.5$$

This equation will coincide with (2.3) if, and only if

$$L = 0,\ M \neq 0,\ N = 0;$$

that is if, and only if

$$(x_{uu},\ x,\ x_u,\ x_v) = 0$$
$$(x_{vv},\ x,\ x_u,\ x_v) = 0$$
$$(x_{uv},\ x,\ x_u,\ x_v) \neq 0.$$

The first and second equation say that x_{uu} and x_{vv} are linear combinations of x, x_u and x_v, whereas the last relation says that x_{uv} is linearly independent of x, x_u and x_v. Hence we have proved that the parametric curves on S_x are asymptotics on S_x if, and only if there exist scalar functions $p(u, v)$, $q(u, v)$, $\alpha(u, v)$, $\beta(u, v)$, $\gamma(u, v)$ and $\delta(u, v)$ such that

$$\begin{aligned} x_{uu} &= px + \alpha x_u + \beta x_v \\ x_{vv} &= qx + \gamma x_u + \delta x_v \end{aligned} \qquad (2.5)$$

and no relation of the form

$$Px_{uv} + Qx_u + Rx_v + S_x = O,$$

where P, Q, R and S are scalar functions of u and v, exists.

Thus we have shown that a surface S_x is an integral surface of a system (2.5) of second order partial differentiations when the parametric curves are the asymptotic curves on the surface. But it is not true conversely that given a system (2.5) there exists an integral surface S_x. For the existence of S_x satisfying (2.5), the system (2.5) must have to be completely integrable. The coefficients p, q, α, β, γ, δ cannot be arbitrary. They must satisfy certain conditions for integrability. These conditions arise from the demand that

$$(x_{uu})_{uv} = (x_{uu})_{vu}, \quad (v_{uu})_{vv} = (x_{vv})_{uu}, \quad (x_{vv})_{uv} = (x_{vv})_{vu}.$$

Lane proved that S_x is an integral of the system

$$\begin{aligned} x_{uu} &= px + \theta_u\, x_u + \beta x_v \\ x_{vv} &= qx + \gamma\, x_u + \theta_v x_v \end{aligned} \qquad (2.6)$$

(1)

provided the following integrability conditions are satisfied:

$$\begin{aligned} \theta_{uvv} &= (\varphi\gamma)_u + 2q_u + \theta_v\,\theta_{uv} - \beta\gamma\psi \\ \theta_{uuv} &= (\varphi\beta)_v + 2p_v + \theta_u\,\theta_{uv} - \beta\gamma\varphi \\ p_{vv} - \theta_v p_v + \beta q_v + 2q\beta_v &= q_{u\bar{u}}\ \theta_u q_u + \gamma p_u + 2p\gamma_u \end{aligned} \qquad (2.7)$$

where

$$\varphi = (\log \beta\gamma^2)_u, \quad \psi = (\log\ \beta^2\gamma)_v \qquad (2.8)$$

When the integrability conditions (2.7) are satisfied by the coefficients of (2.6), the system is completely integrable and has four independent solutions:

$$x_i = x_i(u, v), \quad i = 1, 2, 3, 4$$

which are then the coordinates of a variable point P_x the locus of which is a surface S_x.

Thus the systems (2.6) and (2.7) are differential defining an analytic surface S_x referred to its asymptotic net.

A second net of curves on the surface to which S_x may be referred is a conjugate net. In a conjugate net, the tangents of the

(1) E. P. Lane, Projective Differential Geometry of Curves and Surface, University of Chicago Press, 1932 p. 69. This book will be referred to as Lane.

curves of one family at the points of each fixed curve of the other family form a developable. If the conjugate net is taken as the reference curves on the surface S_x, then S_x is an integral of a system of the form

$$x_{uu} = ax_{vv} + bx_u + cx_v + dx \qquad (1)$$

$$x_{uv} = b'x_u + c'x_v + d'x,$$

with the corresponding integrability conditions.

Congruences of Lines

A congruence of lines is a two-parameter family of lines. It is the locus of a line moving with two degrees of freedom. A congruence may be formed in several ways. For our purposes, we shall consider congruence whose generators are in 1-1 correspondence with the points of a surface.

Let S_x and S_y be two analytic surfaces whose points are in 1-1 correspondence. Then the points P_x on S_x and P_y on S_y correspond if they have the same curvillinear coordinates (u, v). The line P_x P_y generates a congruence as P_x and P_y vary independently over S_x and S_y respectively, since each line P_xP_y is determined by two independent parameters u and v.

Consider now two congruences $\Gamma_{xy} = (P_xP_y)$ and $\Gamma_{\bar{x}\underline{y}} = (P_{\bar{x}}P_{\underline{y}})$ determined by the pairs (S_x, S_y) and $(S_{\bar{x}}, S_{\underline{y}})$ of surfaces, whose points are in a 1-1 correspondence; that is, the points P_x, P_y, $P_{\bar{x}}$ and $P_{\underline{y}}$ have the same curvillinear coordinates (u, v). Then the generator P_xP_y of Γ_{xy} and the generator $P_{\bar{x}}P_{\underline{y}}$ of $\Gamma_{\bar{x}\underline{y}}$ are corresponding generators of Γ_{xy} and $\Gamma_{\bar{x}\underline{y}}$.

The aim of this paper is to study this congruence correspondence. We shall start by setting up a system of partial differential equations for the congruences Γ_{xy} and $\Gamma_{\bar{x}\underline{y}}$ and deduce therefrom the properties of the new geometric configuration.

We shall assume that P_y does not lie in the tangent plane of S_x at P_x. Then $(x, x_u, x_v, y) \neq 0$. Therefore, if α and β are any vector function of u and v, there exist scalar functions A, B, C, D, A′, G′, D′, of u and v such that

$$\alpha = Ax + By + Cx_u + Dx_v$$

$$\beta = A'x + B'y + C'x_u + D\ x_v.$$

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(1) E. J. Wilczynski, Projective Differential Geometry of Curved Surfaces, I Trans. of the Am. Math. Society 8, 233-60.

Eliminating x_u and x_v in succession, we have:

$$\frac{D'\alpha - D\beta}{CD' - c'D} = \frac{AD' - A'D}{CD' - C'D}x + \frac{BD' - B'D}{CD' - C'D}y + x_u$$

$$\frac{C'\alpha - c\beta}{C'D - c'D'} = \frac{Ac' - A'C}{C'D - cD'}x + \frac{BC' - B'C}{C'D - cD'}y + x_v,$$

$$CD' - C'D \neq O.$$

Then if we put

$$\bar{X} = \frac{D'\alpha - D\beta}{CD' - C'D}, \quad \underline{Y} = \frac{C'\alpha - c\beta}{CD' - C'D}$$

we can write the above equations in the form

$$\begin{aligned} \bar{X} &= x_u - lx - my \\ \underline{Y} &= x_v - l'x - m'y. \end{aligned} \qquad (3.1)$$

These define the surfaces S_x and S_y. Note that if $(x, y, x_u\, x_v) \neq 0$ then

$$(\bar{x}, \underline{y}, x, y) \neq 0.$$

Hence every vector function may be expressed as a linear combination of x, y, $\bar{x}$ and $\underline{y}$: Therefore, the study of a pair, of congruences whose generators are in 1-1 correspondence may be based on the following system of linear homogeneous partial differential equations:

Differential Equations for Γ_{xy} and $\Gamma_{\bar{x}\underline{y}}$

$$\begin{aligned} x_u &= lx + my + \bar{x} & \qquad y_u &= ax + by + c\bar{x} + d\underline{y} \\ x_v &= l'x + n'y + \underline{y} & y_v &= a'x + b'y + c'\bar{x} + d'y \\ \bar{x}_u &= e\,x + fy + g\bar{x} + h\underline{y} & \underline{y}_u &= px + qy + r\bar{x} + s\underline{y} \\ \bar{x}_v &= e'x + f'y + g'\bar{x} + h'y & \underline{y}_v &= p'x + q'y + r'\bar{x} + s'\underline{y} \end{aligned} \qquad (3.2)$$

The scalar coefficients in the above equations are not arbitrary functions of u and v but must satisfy the following integrability conditions which are obtained by imposing the conditions $x_{uv} = x_{vu}$, $y_{uv} = y_{vu}$, etc.

Integrability Conditions

$$l'u + p + m'a = lv + e' + ma'$$

$$m'_u + q + l'm + m'b = m_v + f' + l\,m' + m\,b'$$

$$r + l' + m'c = g' + mc'$$

$$s + m'd = h' + l + md'$$

$$a'_u + a'l + ab' + c'e + d'p = a_v + al' + a'b + ce' + dp'$$

$$b'_u + a'm + c'f + d'q = b_v + am' + cf' + dq'$$

$$c'_u + a' + b'c + c'q + d'r = C_v + bc' + cg' + dr' \qquad (3.3)$$

$$d_u' + b'd + c'b + d's = d_v + bd' + ch' + ds' + a$$

$$e_u' + e'l + f'a + g'e + h'p = e_v + el' + fa' + ge' + hp'$$

$$f_u' + e'm + f'b + g'f + h'q = f_v + em' + fb' + fg'\ hg'$$

$$g'_u + e' + f'c + h'r = g_v + fc' + hr'$$

$$h'_u + f'd + g'h + h's = h_v + e + fd' + gh' + hs'$$

$$p'_u + p'l + q'a + r'e + s'p = p_v + pl' + qa' + re' + sp'$$

$$q'_u + p'm + q'b + r'f + s'q = q_v + pm' + qb' + rf' + sq'$$

$$r'_u + p' + q'c + r'g + s'r = r_v + qc' + rg' + sr'$$

$$s'_u + q'd + r'h = s_v + p + qd' + rh'.$$

We now introduce an additional relation between Γ_{xy} and $\Gamma_{\underline{\bar{x}}\underline{y}}$.

Definition: Suppose that each generator P_xP_y of Γ_{xy} intersects an infinite number of surfaces in such a way that the tangent plane of each surface at its point of intersection with P_xP_y pass thru the corresponding line $P_{\underline{\bar{x}}}P_{\underline{y}}$ of $\Gamma_{\underline{\bar{x}}\underline{y}}$. Then the congruence $\Gamma_{\underline{\bar{x}}y}$ is said to be *stratifiable* with Γ_{xy}.

This definition is due to Fubini.[1]

We now raise the fundamental problem: Given a congruence Γ_{xy}, can we construct a second congruence $\Gamma_{\underline{\bar{x}}\underline{y}}$ so that it shall be stratifiable with Γ_{xy}?

To answer this question, let a surface S_z, intersect the generator P_xP_y of Γ_{xy} at P_z, where, $z = x + \lambda y$, and inquire whether the function $\lambda\ (u, v)$ can be determined so that the tangent plane of S_z at P_z will pass thru the line $P_{\underline{\bar{x}}}P_{\underline{y}}$.

Let

$$(W, \underline{\bar{X}}, \underline{Y}, \qquad x + \lambda y) = 0$$

(1) M. S. Finikoff, Sur les congruences stratifiables, Circolo Matematica di Palermo, t. 53, p. 313. This paper will be referred to as Finikoff.

be a plane thru $\underline{\bar{X}}$, Y and $Z = x + \lambda y$. Then this will coincide with the tangent plane of S_z at P_z, where $z = x + \lambda y$ if, and only if at W $= x + \lambda y$, the following equations are satisfied:

$$(W, \underline{\bar{X}}, \underline{Y}, \; x + \lambda y) = 0$$

$$(W, \underline{\bar{X}}, \underline{Y}, \; x + \lambda y)_u = 0$$

$$(W, \underline{\bar{X}}, \underline{Y}, \; x + \lambda y)_v = 0$$

The first equation is identically satisfied. The last reduces to

$$(x + \lambda y \;, \underline{\bar{x}}, \underline{y}, \;\; x_u + \lambda y_u + \lambda_u y) = 0$$

$$(x + \lambda y \;, \underline{\bar{x}}, \underline{y}, \;\; x_v + \lambda y_v + \lambda_v y) = 0$$

which can be further simplified into

$$(x, y, \underline{\bar{x}}, y) \, [\lambda u - a\lambda^2 - (1 - b) \, \lambda + m] = 0$$

$$(x, y, \underline{\bar{x}}, \underline{y}) \, [v - a'\lambda^2 - (1' - b') \, \lambda + m'] = 0 \; ,$$

where the first four equations for x_u, x_v, y_u, y_v in (3.2) have been used. Since $(x, y, x, y) \neq 0$, we conclude that

$$\begin{aligned} &\lambda_u - a \, \lambda^2 - (1 - b) \, \lambda + m =) \; 0 \\ &\lambda_v - a' \, \lambda^2 - (1' - b') \, \lambda \; + m' = 0 \end{aligned} \tag{3.4}$$

which are completely integrable provided the following integrability conditions are satisfied:

$$\begin{aligned} &a'_u + a'(1-b) = a_v + a(1' - b') \\ &l'_u - b'_u - 2a'm = l_v - b_v - 2am' \\ &m'_u + m(l' - b') = m_v + m' \, (1-b) \end{aligned} \tag{3.5}$$

When these conditions (3.5) are satisfied, the equations (3.4) have an infinite number of solutions for λ. For each we get a surface S_z, $z = x + \lambda y$, whose tangent plane at P_z passes thru the line $P_x P_y$ of the congruence Γ_{xy}. In other words the generator $P_{\bar{x}} P_y$ is the axis of a pencil of planes which are tangent planes of the family $\sigma = (S_z)$ of surfaces at their points of intersection with the generator $P_x P_y$ of Γ_{xy}.

Let us note that the required congruence $\Gamma_{\bar{x}y}$ in the fundamental problem is determined by the vector functions $\underline{\bar{x}}$ and $\underline{y}$ defined in (3.1) which in turn are determined by the functions $\bar{1}$, $1'$, m and m', which must satisfy (3.5) in order that $\Gamma_{\underline{\bar{x}y}}$ will be stratifiable with Γ_{xy}. How many such congruences $\Gamma_{\underline{\bar{x}y}}$ are there? This is answered by (3.5). Regarded as a system for the determination of l, l', m, m' the system (3.5) consists of three operations in four unknowns. Hence there is an infinite solution for l, l', m and m'. Therefore there are an infinite number of congruences $\Gamma_{\underline{\bar{x}}y}$ stratifiable with a given congruence Γ_{xy}. We have proved the following theorem:

Theorem. A given congruence Γ_{xy} determines infinitely many congruences $\Gamma_{\underline{\bar{x}}y}$ simply stratifiable with Γ_{xy}.

In the next section we shall consider the conditions under which the stratifiability relation between Γ_{xy} and $\Gamma_{\underline{\bar{x}}y}$ is symmetrical. What further conditions on the coefficients of (3.2) must be imposed so that Γ_{xy} is stratifiably with $\Gamma_{\underline{\bar{x}}y}$? Is it possible to determine a function $\mu(u, v)$ so that the tangent plane to S_z, $Z = \underline{\bar{x}} + \mu \underline{y}$, at P_z shall pass thru the generator $P_x P_y$? If the answers are affirmative then Γ_{xy} and $\Gamma_{\underline{\bar{x}}y}$ are said to be *doubly* stratifiable.

Doubly Stratifiable Congruences.

To answer these questions, consider the plane determined by the points P_x, P_y and P_z. The equation of this plane is

$$(w, x, y, z) = 0,$$

This plane will coincide with the tangent plane to S_z at P_z if and only if

$$(w, x, y, Z) = 0$$

$$(w, x, y, Z)_u = 0$$

$$(w, x, y, Z)_v = 0$$

are satisfied for $W = Z - \underline{\bar{x}} + \mu \underline{y}$. By an argument similar to what led us to equations (3.4) we obtain the system:

$$\begin{aligned} &\mu_u - r\mu^2 + (s-g)\mu + h = 0 \\ &\mu_u - r'\mu^2 + (s'-g')\mu + h' = 0, \end{aligned} \qquad (4.1)$$

with the following conditions for integrability:

$$\begin{aligned} &r'_u + r'(g-s) = r_v + r(g'-s') \qquad (4.2) \\ &g'_v - s'_u - 2r'h = g_v - s_v - 2r h' \\ &h'_u + h(g'-s') = h_v + h'(g-s) \end{aligned}$$

When these conditions are satisfied then (4.1) has an infinite number of solutions for μ (u, v). Therefore there are infinite number of surfaces S_z, where $Z = \underline{\bar{x}} + \mu \underline{y}$. The totality of these surfaces will be denoted by $\Sigma = [S_z]$.

We now apply the foregoing theory to an important particular pair of congruences associated with a given surface. They are called the reciprocal congruences of Green.

Surfaces with Doubly Stratifiable
Reciprocal Congruences

Let S_x be a surface on which the u-curve and the v-curve are the asymptotic curves on the surface. Then from (2.6), (2.7), (2.8) S_x is an integral surface of the following pair of partial differential operations:

$$
\begin{aligned}
y_{uu} &= \bar{\phi}\, x + \theta_u\, x_u + \beta\, x_v \\
x_{vv} &= \bar{q} x + \gamma\, x_u + \theta_v\, x_v, \qquad\qquad (5.1)\\
\theta &= \log \beta \gamma,
\end{aligned}
$$

with the following integrability conditions

$$
\begin{aligned}
\theta_{uvv} &= (\gamma\varphi)_u + 2\bar{q}_u + \theta_v \theta_{uv} - \beta\gamma\psi \\
\theta_{uuv} &= (\beta\psi)_v + 2\bar{\phi}_v + \theta_u \theta_{uv} - \beta\gamma\psi \qquad\qquad (5.2)
\end{aligned}
$$

$$\bar{\phi}_{vv} - \theta_v\, \bar{\phi}_v + \beta\, \bar{q}_v + 2\bar{q}\,\beta_v = \bar{q}_{uu} - \theta_u\, \bar{q}_u + \gamma\,\bar{\phi}_u + 2\bar{\phi}\gamma_u,$$

where $\varphi = (\log \beta\gamma^2)_u$, $\psi = (\log \beta^2\gamma)_v$. (5.3)

We pause to define some terms that will be important in subsequent discussions.

First, the concept of a developable surface; it is the locus of the tangent lines of a curve. The tangents are the generators and the curve is the edge of regression of the developable.

It can be shown that s surface S_x is a developable surface if, and only if,

$$LN - M^2 = 0$$ (1)

where L, M, N are defined in (2.4).

Second, the generators of a congruence can be assembled into a one-parameter family of developable surfaces in two ways. The lines of a congruence are the common tangents to two surfaces, called the *focal surfaces* of the congruence. The two points at which a generator touches the *focal surfaces* are called the *focal points* of the generator, and the tangent planes are the *focal planes* of the generator(2)

(1) Lane, p. 38
(2) Ibid, p. 82

The method of obtaining these points and planes of the congruence will be indicated presently in this section.

On a surface S_x referred to the asymptotic net on the surfaces, consider a point P_x and a line 1, thru P_x but not lying in the tangent plane of S_x at P_x. Such a line may be determined by the points P_x and P_y, where

$$y = x_{uv} - \bar{a}x_u - \bar{b}x_v \tag{5.4}$$

where $\bar{a}$ and $\bar{b}$ are scalar functions of u and v. The line 1, at once determines another line 1_2 — the polar reciprocal of 1_1 with respect to the quadric of Lie at the point P_x. Referred to the tetrahedrom $P_x, P_{x_u}, P_{x_v}, P_{x_{uv}}$ the equation of this quadric is

$$2(x_2 x_3 - x_1 x_4) - (\beta\gamma + \theta_{uv}) x_4^2 = 0. \tag{5.5}$$

The polar reciprocal of 1_1 is the line 1_2 which join P_ρ and P_σ

where

$$\rho = x_u - \bar{b}x, \quad \sigma = x_v - \bar{a}x \tag{5.6}$$

As u and v vary, 1_1 and 1_2 generate two congruences Γ_1, and Γ_2 whose generators are in 1-1 correspondence.

Let us find the focal points of the generator 1 and the developables of Γ_1. Let P_z be any point on 1_1 were

$$z = y + \lambda x.$$

Let us determine λ so that P_z will be a focal point[1] 1. Let C be a curve on S_x passing thru P_x. If as P_x varies C, generates a developable of Γ_1 and if P_z is a focal point of 1_1, then P_z will describe of curve of which 1_1, is tangent at P_z. Then

$$\frac{dz'}{dt} = z_u \frac{du}{dt} + z_v \frac{du}{dt}$$

must be a linear combination of x and y only. But we have

$$z_u = (\bar{\phi}_v - \bar{a}\bar{p} + \beta\bar{q} + \lambda u)x + (A + \lambda)x_u + (F - 2\bar{a}\ \beta + \beta\varphi)\, x_v$$
$$- (\bar{b} - \theta_u)y$$

$$z_v = (\bar{q}_u - \bar{b}\bar{q} + \gamma\bar{\phi} + \lambda\ v)x + (G\text{-}2\bar{b}\,\gamma + \gamma\psi)x_u + (\beta + x_v -$$
$$(\bar{a} - \theta_v)y$$

where

$$A = -\bar{a}_u - \bar{a}\bar{b} + \beta\gamma + \theta_{uv},\ F = \bar{\phi} - \bar{b}_u + \bar{b}\,\theta_u - \bar{b}^2 + \bar{a}\,\beta$$

$$B = \bar{b}_v - \bar{a}\,\bar{b} + \beta\gamma + \theta_{uv},\ G = \bar{q} - a_v - \bar{a}\theta_v - \bar{a}^2 + \bar{b}\,\beta.$$

Therefore

$$\frac{dz'}{dt} = P\,x + Qy + [(A + \lambda)\frac{du}{dt} + (G\text{-}2\bar{b} + \gamma\varphi)\frac{dv}{dt}]\ x_u$$

$$+ [\,(p - 2\bar{a}\,\beta + \beta\varphi)\frac{du}{dt}\ (\beta + \lambda)\frac{dv}{dt}]\ x_v$$

where P and Q are functions we shall have no occasion to use. Hence $\frac{dz}{dt}$ is a linear combination of x and y if, and only if

$$\begin{aligned}(A + \lambda)\,du + (G\text{-}2\bar{b}\,\gamma + \gamma\psi)\,dv &= 0\\ (F - 2\bar{a}\,\beta + \beta\psi)du + (\beta + \lambda)\,dv &= 0\end{aligned} \qquad (5.8)$$

Eliminating du and dv from these equations, we have:

$$\begin{vmatrix} A + \lambda & F - 2\bar{a}\beta + \beta \\ F - 2\bar{a}\ \beta + \beta\psi & B + \lambda \end{vmatrix} = 0 \qquad (5.9)$$

or $\lambda^2 + (A + b)\lambda + AB - (F\text{-}2\bar{a}\beta + \beta\psi)\ (G - 2\bar{b}\gamma + \gamma\varphi) = 0.$

If λ_1 and λ_2 are the roots of these quadratic equations then the points P_{z_1} and P_{z_2} where

$$z_1 = y + \lambda_1\,x,\ z_2 = y + \lambda_2 x\ ,$$

are the focal points of 1_1 and the surfaces S_{z_1} nd S_{z_2} are the focal surfaces of Γ_1.

If on the other hand we eliminate λ from (5.8), we obtain

$$\begin{vmatrix} du & Adu + (G - 2\bar{b}\ \gamma + \gamma\varphi)dv \\ dv & (F - 2\bar{a}\ \beta + \beta\varphi)\ du + \beta\,dv \end{vmatrix} = 0 \qquad (5.10)$$

or

$$(F - 2\bar{a}\beta + B\psi)\,du^2 - (\bar{b}_v - a_u)\,dudv - (G - 2\,\bar{b}\,\gamma + \gamma\varphi)dv^2 = 0.$$

This is the differential equation of the curves in which the developables of Γ_1 intersect S_x which will be called the Γ_1-curves.

Since the equation is quadratic, the lines of the congruence can be assembled in two ways. Hence the line 1_1 belongs to two different developables of Γ_1.

Let us similarly find the focal points and developables of Γ_2. The line 1_2 which joins $P\rho$ and $P\sigma$ has a focal point at $\xi = \rho + \mu\sigma$ if

$$\frac{d\xi}{dt} = (\rho_u + \mu_u + \mu\sigma_u)\frac{du}{dt} + (\rho_v + \mu_v\sigma + \mu\sigma_v)dv$$

is a linear combination of ρ and σ only. The values of ρ_u, ρ_v, σ_u and σ_v are given by

$$\rho_u = Fx - (\bar{b} - \theta_u)\rho + \beta\sigma, \ \rho_v = (\bar{b}_v = + \bar{a}\,\bar{b})x - \bar{b} + x_{uv}$$

$$\sigma_u = Gx + \delta\rho - (\bar{a} - \theta_v)\sigma, \ \sigma_v = -(\bar{a}_u + \bar{a}\,\bar{b})x - a\rho + x_{uv}$$

Hence

$$\frac{d\xi}{dt} = P\rho + Q\sigma + [F - \mu(\bar{a}_u + \bar{a}\,\bar{b})]\ du + [-\bar{b}_v - \bar{a}\,\bar{b} + \mu G]\ dv\ x$$

$$+ [\mu d_u + d_v]\ x_{uv}$$

will be a linear combination of ρ and σ if, and only if

$$[F - \mu(\bar{a}_u + \bar{a}\,\bar{b})]\ du + [-\bar{b}_v - \bar{a}\,\bar{b} + \mu G]\ dv = 0$$

$$\mu\, du + dv = 0 \qquad (5.12)$$

Eliminating du and dv, from (5.12), we have:

$$\begin{matrix} F - \mu(\bar{a}_u + \bar{a}\,\bar{b}) & -\bar{b}_v - \bar{a}\,\bar{b} + \mu G \\ \mu & 1 \end{matrix} = 0 \qquad (5.13)$$

or $\quad F + (\bar{b}_v - \bar{a}_u)\mu - G\mu^2 = 0$

If μ_1 and μ_2 are the roots of this quadratic then the focal points of 1_2 are $P_{\xi 1}$ and $P_{\xi 2}$ where,

$$\xi_1 = \rho + \mu_1\sigma, \quad \xi_2 = \rho + \mu_2\sigma$$

and the loci of $P_{\xi 1}$ and $P_{\xi 2}$ are the focal surfaces of $\Gamma 2$.

Eliminating μ, from (5.12) we have

$$\frac{F du - (\bar{b}_v + \bar{a}\,\bar{b})dv}{dv} - \frac{(\bar{a}_u + \bar{a}\,\bar{b})\,du + G\,dv}{du} = 0 \qquad (5.14)$$

or

$$F\ du^2 - (\bar{b}_v - \bar{a}u)\ dudv - G\ dv^2 = 0.$$

This is the differential equation of the curves in which the developables of Γ_2 cut the surface S_x. Since the equation is quadratic the developables of Γ_2 can be assembled in two ways.

We now raise a number of questions. Is it possible for Γ_1 and Γ_2 to be doubly stratifiable? What kind of surfaces sustain reciprocal congruences that are doubly stratifiable? What kind of reciprocal congruences, with respect to a surface, are doubly stratifiable?

To answer these questions let us first set up the system (3.2) for the pair Γ_1 and Γ_2. Here ρ and σ will replace $\bar{\underline{x}}$ and $\underline{y}$ in (3.2).

The Equations for Γ_1 and Γ_2

$$x_u = lx + my + \rho \qquad y_u = ax + by + c\,\rho + d\sigma$$

$$x_v = l'x + m'y + \sigma \qquad y_v = a'x + b'y + c'\rho + d'\sigma$$

$$\rho_u = ex + fy + g\rho + h\sigma \qquad \sigma_u = px + qy + r\,\rho + s\sigma$$

$$\rho_v = e'x + f'y + g'\rho + h'\sigma \qquad \sigma_v = p'x + q'y + r'\,\rho + s'\sigma$$

where

$$a = \bar{p}_v + \beta\bar{q} + \bar{b}\,\theta_{uv} + \bar{b}\gamma\beta - \bar{a}\,\bar{b}^2 - \bar{a}_u\bar{b} + \bar{a}\,\beta_v + \bar{a}\,\beta\theta_v -$$
$$\bar{a}\beta - \bar{a}b_u - \bar{a}\ \bar{b}^2$$

$$a' = \bar{q}_u + \bar{p}\gamma + \bar{b}\gamma_u + \bar{b}\ \gamma\ \theta_u - \bar{b}^2\gamma + \bar{a}\,\bar{b}\,\theta_v - \bar{a}^2\,\bar{b}$$

$$b = \theta_u - \bar{b},\ c = \theta_{uv} + \beta\gamma - \bar{a}\ \bar{b} - \bar{a}_u,\ d = \bar{p} + \beta_v + \beta\theta_v -$$
$$\bar{a}\beta - b_u + \bar{b}\,\theta_u - \bar{b}^2$$

$$b' = \theta_v - \bar{a},\ c' = \bar{q} + \gamma_u + \gamma\theta_u - \bar{b}\gamma + \bar{a}\ \theta_v - a^2,\ d' = \theta_{uv} +$$
$$\beta\gamma - \bar{b}_v - \bar{a}\ \bar{b}$$

$$l = \bar{b},\ \ m = 0,\ l' = \bar{a},\ m' = 0$$

$$e = F = \bar{p} + \theta_u\bar{b} + \bar{a}\beta - \bar{b}_u - \bar{b}_2,\ f = 0,\ g = \theta_u - \bar{b},\ h = \beta$$

$$e' = \bar{a}\bar{b} - \bar{b}_v,\ f' = 1,\ \ g' = \bar{a},\ \ h' = 0$$

$$p = \bar{a}\,\bar{b} - \bar{a}_u,\ q = 1,\ r = 0,\ s = \bar{b}$$

$$p' = G = \bar{q} - \bar{a}_v + \bar{b}\gamma - \bar{a}^2 + \bar{a}\ \beta_v,\ q' = 0,\ r' = \gamma,\ s' = \theta_v - a$$

Imposing conditions (3.5) and (4.2) for double stratifiability we get:

$$a'_u + a'(2\bar{b} - \theta_u) = a_v + a(2\bar{a} - \theta_v)$$

$$\begin{aligned} &\bar{a}_u - \bar{b}_v = 0 \\ &\beta_v = \beta(2\bar{a} - \theta_v) \\ &\gamma_u = \gamma(2\bar{b} - \theta_u) \\ &\bar{a}_u + \bar{b}_v = \beta\gamma + \theta_{uv} \end{aligned} \tag{5.15}$$

From the third and fourth of these equations, we obtain

$$\begin{aligned} \bar{a} &= \frac{\beta_v + \theta_{uv}}{2\beta} = \frac{\psi}{2} \\ \bar{b} &= \frac{\gamma_u + \theta_{uv}}{2\gamma} = \frac{\psi}{2} \end{aligned} \tag{5.16}$$

From (5.16) it follows that l_1 and l_2 are the directrices of Wilczynski and the congruences Γ_1 and Γ_2 are the directrix congruences of Wilczynski.

Inspection of (5.10), (5.14), the second equation of (5.15) and (5.16) shows that the developables of Γ_1 and Γ_2 correspond to the same conjugate net

$$F\ du^2 - G\ dv^2 = 0 \tag{5.17}$$

on S_x.

From the second and last equation of (5.16), we obtain

$$\bar{a}_u = \frac{\beta\gamma + \theta_{uv}}{2}$$

$$\bar{b}_v = \frac{\beta\gamma + \theta_{uv}}{2}$$

But from (5.16) we have

$$\bar{a}_u = \frac{1}{2}\left(\frac{\beta_v + \beta\theta_v}{\beta}\right)_u = \frac{1}{2}\left[(\log\beta)_{vu} + \theta_{vu}\right]$$

$$\bar{b}_v = \frac{1}{2}\left(\frac{\gamma_u + \gamma\theta_u}{\gamma}\right)_v = \frac{1}{2}\left[(\log\gamma)_{uv} + \theta_{uv}\right].$$

Hence $(\log\beta)_{uv} + \theta_{uv} = \beta\gamma + \theta_{uv}$

$(\log\gamma)_{uv} + \theta_{uv} = \beta\gamma + \theta_{uv}.$

Consequently,

$$\frac{\partial_2}{\partial u \partial v}(\log \beta) = \beta\gamma$$
$$\frac{\partial^2}{\partial u \partial v}(\log \gamma) = \beta\gamma \tag{5.18}$$

Therefore,

$$\frac{\partial^2}{\partial u \partial v}\left(\log \frac{\beta}{\gamma}\right) = 0 \tag{5.19}$$

Equations (5.18) state that the asymptotic tangents of S_X belong to two linear complexes — the osculating linear complexes of the asymptotics.
The equations of these complexes are

$$2w_{23} - \varphi\, w_{42} - \psi w_{34} = 0$$
$$2w_{14} - \varphi\, w_{42} + \varphi\, w_{34} = \theta \tag{5.20}$$

They form a pencil of complexes whose directrix congruences are Γ_1 and Γ_2.

Consider the most general transformation which leaves the form of equations (5.1) invariant, namely

$$x = c\,\bar{x}, \quad \bar{u} = \lambda(u), \quad \bar{v} = \mu(v), \quad c \text{ const} \tag{5.21}$$

where λ (u) is a function of u alone and μ (v), a function of v alone.
Then the new coefficients of (5.1) are given by

$$P = \frac{\bar{p}}{\lambda'^2(u)}, \quad Q = \frac{\bar{q}}{\mu^2(v)}$$
$$\beta = \beta\frac{\mu'(v)}{\lambda'^2(u)}, \quad \gamma = \frac{\gamma\lambda'(u)}{\mu'^2(v)}$$
$$\theta = \theta - \log \lambda'(u)\ \mu'(v).$$

It is also easy to show that $\theta = \log \bar{\beta}\,\bar{\gamma}$.

$$\frac{\beta}{\gamma} = \frac{V(v)}{U(u)} \tag{5.22}$$

where U(u) is an arbitrary function of u alone and V(v) is an arbitrary function of v alone. Let us choose λ(u) and μ(v) so that

$$\frac{\bar{\beta}}{\bar{\gamma}} = 1.$$

To do this, we note that,

$$\frac{\bar\beta}{\bar\gamma} = \frac{\mu'^3(v)}{\lambda'^3(u)} \cdot \frac{\beta}{\gamma} = \frac{\mu'^3(v)}{\lambda'^3(v)} \cdot \frac{U(u)}{V(v)}$$

Hence $\frac{\bar\beta}{\bar\gamma} = 1$ if we choose λ (u) and μ (v) so that

$$\lambda'^3(u)\ U(u) = k$$

$$\mu^{3\prime}(v)\ V(v) = k$$

where $k \neq 0$ may be taken equal to 1. Then we may choose

$$\lambda(u) = \frac{du}{\sqrt{U(u)}}, \quad \mu(v) = \frac{dv}{\sqrt[3]{V(v)}}$$

For this choice of λ (u) and $\mu(v)$, the transformation (5.21) transforms the ratio $\frac{\beta}{\gamma}$ into

$$\frac{\bar\beta}{\bar\gamma} = 1.$$

Assuming that this transformation has been carried out we then have a system (5.1) in which $\beta = Y$ and (5.18) becomes

$$\frac{\partial\ (\log \beta)}{\partial u \partial v} = \beta^2 \tag{5.22}$$

the general solution of which is

$$\beta = \frac{\sqrt{U'v'}}{(U + V)} \tag{5.23}$$

where U(u) and V(v) are arbitrary functions of u alone and v alone respectively. Substituting $\beta = Y = \frac{\sqrt{U'V'}}{U+V}$ in the integrability conditions (5.2) and solving for $\bar p$ and $\bar q$, we get

$$\bar p = \frac{3}{2} \left(\frac{\sqrt{U'V'}}{U+V}\right)_v - \frac{3}{4} \left(\frac{U'}{U+V}\right)_v + A(u)$$

$$\bar q = -\frac{3}{2} \left(\frac{\sqrt{U'V'}}{U+V}\right)_u - \frac{3}{4} \left(\frac{V'}{U+V}\right)_u + 3(v)$$

where A (u) and B (v) are arbitrary functions.

By (5.16), (5.23) and (5.24) the functions β, $\bar p$, $\bar q$, $\bar a$ and $\bar b$ have been computed to satisfy all integrability conditions (5.15) except the first. These five functions β, $\bar{p}$, $\bar{q}$, $\bar{a}$ and $\bar{b}$ depend on four arbitrary functions U(u), V(v), A(u) and B(v). If these four arbitrary functions are now chosen so that the two remaining conditions — the last equation of (5.2) and the first equation of (5.15) — are satisfied, then the five functions β, $\bar{p}$, $\bar{q}$, $\bar{a}$ and $\bar{b}$ determine a surface S_x which sustain a doubly stratifiable pair of reciprocal congruences Γ_1 and Γ_2. Since there are four functions to be chosen and only two equations to satisfy, the choice of the arbitrary set U(u), V(v), A(u), B(v) can, clearly, be made in an infinite number of ways.

We now recapitulate the foregoing results in the following theorem.

Theorem. There is an infinite number of surfaces which sustain a pair of doubly stratifiable reciprocal congruences. These congruences are the directrix congruences of Wilczynski whose developables correspond on the surface to the same conjugate net. The asymptotic tangents of the surface belong to the linear complexes which give rise to the directrix congruences of Wilczynski.

We now consider a particular solution $\beta = \dfrac{1}{u+v}$ of (5.22) which leads to an interesting result.

6. *The solution* $\beta = \gamma = \dfrac{1}{u+v}$

The particular solution

$$\beta = \frac{1}{u+v} \tag{6.1}$$

of the equation

$$\frac{\partial^2 (\log \beta)}{\partial_u \, \partial_v} = \beta^2$$

gives
$$\bar{p} = \bar{q} = \frac{9}{4}\beta^2$$
$$\bar{a} = \bar{b} = \frac{3}{2}\beta. \tag{6.2}$$

The equations (5.1) and (5.14) of the developables Γ_1 and Γ_2 are satisfied identically. Hence the developables are indeterminate.

The line l_1 of Γ_1 is now determined by the points P_x and P_y where

$$y = x_{uv} + \left(\frac{3}{2}\beta\right) x_u + \left(\frac{3}{2}\beta\right) x_v$$

The point P_z, where

$$z = y + (\frac{3}{4}\beta^2)x = (\frac{3}{4}\beta^2)x + (\frac{3}{2}\beta)x_u + (\frac{3}{2}\beta)x_v + x_{uv},$$

lies on 1_1; its coordinates, referred to the tetrahedron

$(P_x, P_{xu}, P_{xv}, P_{x_{uv}})$,

are:

$$kx_1 = \frac{3}{4}\beta^2,\ kx_2 = \frac{3}{2}\beta,\ kx_3 = \frac{3}{2}\beta,\ kx_4 = 1. \qquad (6.3)$$

The line 1_2 of Γ_2 is determined by P_ρ and P_σ, where

$$\rho = x_u + (\frac{3}{2}\beta)x, \quad \sigma = x_v + (\frac{3}{2}\beta)x.$$

Referred to the same tetrahedron, the coordinates of P_ρ and P_σ are

$$(\frac{3}{2}\beta, 1, 0, 0), \ (\frac{3}{2}\beta, 0, 1, 0).$$

Obviously P_ρ and P_σ lie in the plane π whose equation is:

$$\pi:\ x_1 - (\frac{3\beta}{2})x_2 - (\frac{3\beta}{2})x_3 + (\frac{15\beta^2}{4})x_4 = 0. \qquad (6.4)$$

Therefore the line 1_2 also lies in this plane whose coordinates are

$$k\xi_1 = 1,\ k\xi_2 = \frac{-3\beta}{2},\ k\xi_3 = \frac{-3\beta}{2},\ k\xi_4 = \frac{15\beta^2}{4}. \qquad (6.5)$$

From (6.3) it follows that P_z lies in the plane π; that is 1_1 pierces the plane π at P_z. This makes π and P_z united in position.

If in (6.3) we eliminate k and β homogeneously, we have

$$3x_1x_4 - x_2x_3 = 0 \qquad (6.6)$$

which is quadric.

Similarly, from (6.5) we obtain

$$3\xi_1\xi_4 - 5\xi_2\xi_3 = 0 \qquad (6.7)$$

whose point equation is

$$5x_1x_4 - 3x_2x_3 = 0 \qquad (6.8)$$

which is another quadric.

Both quadrics (6.6) and (6.8) contain the points P_x, P_{x_u}, P_{x_v}, $P_{x_{uv}}$ and both are non-degenerate. The properties of these quadrics will be investigated in another paper which will deal with the surfaces of the congruences.