Algebraically-Closed Groups: An Application of Model Theory to Algebra

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This paper is a short introduction to a method (derived from logic and model theory) of constructing algebraic structures (groups, semigroups, rings, etc.) with given properties. It was first introduced by Abraham Robinson, inspired by Cohen's method of forcing in set theory. We shall show some applications to the theory of algebraically closed groups.

Some Preliminaries on Algebraically Closed Groups

In a 1951 paper [7], W.R. Scott introduced the notion of an algebraically closed group: *A group G is said to be algebraically closed if every consistent finite system of equations, with parameters in G, is solvable in G.* A system of equations is said to be consistent over G, if it has a solution in a group extending G. Scott proved that every group can be extended to an algebraically closed group.

In 1952, B.H. Neumann [5] showed that aside from the trivial group, we get the same class of groups if we replace "finite system of equations" to "finite system of equations and inequations". He also showed that an algebraically closed group is simple.

In 1971, B.H. Neumann [6] considered the isomorphism problem for algebraically closed groups and showed that it would be very difficult to tell two algebraically closed groups apart in the following sense. *Let H be a finitely generated group.* We shall say that H is *recursively absolutely presented* if H is given by a recursively enumerable set of equations and inequations in the generators. Neumann proved: *Any finitely generated group, which is recursively absolutely presentable can be imbedded in any non-trivial algebraically closed group.* This result makes it practically impossible to tell two algebraically closed groups apart by looking at their finitely generated subgroups.

The new results on algebraically closed groups come from applications of A. Robinson's *method of forcing* in Model Theory. The method allows us to construct many non-isomorphic countable algebraically closed groups with specified properties. We shall illustrate the method by giving the proof of the converse to B.H. Neumann's theorem above (due to A. MacIntyre [4]). It will be noted from the proof that the method applies as well to other algebraic structures, such as semigroups, commutative rings, division rings, etc. In fact, there are some very interesting results on the structures of such algebraic structures, satisfying an analogous notion of algebraic closure (Cherlin[1], Hirschfeld and Wheeler [2]).

Forcing in Model Theory and Omitting Types

Definition. A ***condition p*** is a finite consistent set of atomic sentences $\varphi\ (a_1, \ldots, a_n)$ and negated atomic sentences $\neg\varphi\ (a_1, \ldots, a_n)$ (in our case, equations and inequations in $a_1, \ldots, a_n$).

Definition. Given a condition p and a sentence φ, we define "p forces φ" (written $p \Vdash \varphi$) as follows:

(1) If φ is atomic, $p \Vdash \varphi$ iff $\varphi \in p$

(2) If φ is $\varphi_1 \vee \varphi_2$, $p \Vdash \varphi$ iff $p \Vdash \varphi_1$ or $p \Vdash \varphi_2$.

(3) If φ is $\varphi_1 \wedge \varphi_2$, $p \Vdash \varphi$ iff $p \Vdash \varphi_1$ and $p \Vdash \varphi_2$

(4) If φ is $\exists \chi\ \varphi(\chi)$, $p \Vdash \varphi$ iff for some *a* occurring in p, $p \Vdash \varphi(a)$.

(5) If φ is $\neg\varphi$, $p \Vdash \varphi$ iff there is no condition $q \geq p$ such that $p \Vdash \varphi$.

(6) If φ is $\forall \chi \varphi(\chi)$, then $p \Vdash \varphi$ iff $p \Vdash \neg\exists \chi \neg\varphi(\chi)$.

Definition. Let G be a group. G forces a sentence φ iff some condition p true in G forces φ.

G is a ***generic group*** iff for any sentence φ defined in G, φ is true in G iff G forces φ.

The interest of forcing a generic group is as follows: We can construct a sequence of conditions

$$p_0 \subset p_1 \subset p_2 \subset \ldots \subset p_n \subset \ldots$$

in such a way that the union $\bigcup_{n=0}^{\infty} P_n$ is a maximal consistent set of equations and inequations. It determines a unique generic group G. It can be easily shown that generic groups are algebraically closed. By a careful construction of the conditions p_n, we can make G satisfy certain predetermined properties.

To get further refinements on the structure of G, we introduce the notion of "G omitting a type".

Definition. Δ is a quantifier-free n-type if

(i) Δ is a set of basic formulas with free variables $v_0, \ldots, v_{n-1}$ (basic means "atomic or negated atomic")

(ii) Δ is consistent with the axioms for groups

(iii) For any atomic formula $\varphi(v_0, \ldots, v_{n-1})$ either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$.

Definition. A group G realizes a quantifier free n-type if $\ni a_0, \ldots, a_{n-1} \in G$ such that $\varphi(a_0, \ldots, a_{n-1})$ is true in G for all $\varphi \in \Delta$.

We now illustrated the use of forcing and omitting types by giving MacIntyre's proof of the converse of B.H. Neumann's theorem.

Applications of Forcing and Omitting Types

Main Theorem. Let Δ be a quantifier-free n-type, which is non-recursive. Then there exists a generic group G which omits Δ.

Corollary. Let H be a finitely generated group, which is imbeddable in all non-trivial algebraically-closed groups. Then H can be recursively presented with solvable word-problem.

Proof of Corollary. Let $a_0, \ldots, a_{n-1}$ be generators for H and let $\Delta = (Q(v_0, \ldots, v_{n-1}): \varphi(a_0, \ldots, a_{n-1})$ is true in H. Then Δ is a quantifier-free n-type. We claim that Δ is recursive. For suppose otherwise.

Then by the main theorem, there is a generic group G which omits Δ. But then this means that H cannot be imbedded, in G, (if we had an imbedding $a_i \to a_i'$ where $a_i' \in G$, then $a'_0 \ldots, a'_{n-1}$ in G would realize Δ). Since G is generic, it is algebraically closed. We would then have a contradiction to the hypothesis. Finally, Δ recursive clearly implies that H is recursively presented with solvable word problem.

Proof of the Main Theorem. The proof depends on the construction of a sequence of conditions $q_0 \leq q_1 < q_2 \leq \ldots \leq q_m \leq \ldots$ and atomic formulas $\varphi_0, \varphi_1, \ldots, \varphi_m, \ldots$ the following satisfying properties:

Let φ_m be an enumeration of all sentences (in the language of groups with constants $c_0, \ldots, c_n, \ldots$), $\delta_0, \delta_1, \delta_2, \ldots, \delta_m, \ldots$ an enumeration of all possible finite sequences of terms. Then

(1) For each atomic sentence φ, either $\varphi \in \bigcup_m q_m$ or $\neg\varphi \in \bigcup_m q_m$

(2) For each m, either $q_{m+1} \Vdash \varphi_m$ or $q_{m+1} \Vdash \neg\varphi_m$

(3) If $\delta_m = < t_0, \ldots, t_{n-1} >$ then
either (a) $\varphi_m(v_0, \ldots, v_{n-1}) \in \Delta$ and $q_{m+1} \Vdash \neg\varphi_m(t_0, \ldots, t_{n-1})$
or (b) $\neg P_m(v_0, \ldots, v_{n-1}) \in \Delta$ and $q_{m+1} \Vdash \varphi_m(t_0, \ldots, t_{n-1})$

The difficult part is to satisfy condition (3) and it is here that we use the fact that Δ is not recursive. The rest is straightforward.

From $\bigcup_m q_m$, we then get a generic group G. The elements of G are named by terms t and thus for any elements $a_0, \ldots, a_{n-1}$ of G we have from condition (3) that there exists a formula $\varphi(v_0, \ldots, v_{n-1}) \in \Delta$ such that $\varphi(a_0, \ldots, a_{n-1})$ is false in G.

Thus G omits Δ.

Further Results

The method of forcing and omitting types can be used to construct generic groups omitting certain preassigned sets of types. Thus the method gives us a way of studying the structure of algebraically closed groups. Among the more interesting results are:

(1) Every countable algebraically closed group contains a proper copy of itself.

(2) Every countable algebraically closed group has $2^{\aleph_0}$ automorphisms.

The method also applies to other algebraic structure, such as commutative rings, inverse semi-groups, division rings. It constructs generic (rings, semi-groups, division rings), which may be seen to be "algebraically closed" in an appropriate sense.

The results on division rings are particularly interesting. Analogous to the result above on algebraically closed groups, we have that: "Every countable algebraically closed division ring D contains a proper copy of itself." From a recursion-theoretic point of view, the theory of "algebraically- closed" division rings is very complex. More precisely, first-order arithmetic can be interpreted in the theory of "algebraically closed" division rings. The method of forcing we described above is called *finite forcing*, distinguishing it from another method called *infinite forcing*. The structures constructed by infinite forcing are called infinitely generic. If we take the theory of *infinitely generic division rings*, the second-order arithmetic can be interpreted in this theory. The results on division rings depend on a very important theorem of P.M. Cohn: *The theory of division rings has the amalgamation property*, i.e., in the category of division rings and embeddings, any diagram of the following type can always be completed:

$$\begin{tikzcd} & B_1 \arrow[dr, dashed] & \\ A \arrow[ur] \arrow[dr] & & C \\ & B_2 \arrow[ur, dashed] & \end{tikzcd}$$

References

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Heinrich Kuhn, Ph.D., Discussant

I would first like to say that this is clearly a very sophisticated area of mathematics and anybody who is a specialist in this area should serve our highest consideration. It's certainly correct to mention from the Australian Algebraes that this algebraic structures are so complicated that algebraes would not dare to. This would lead to my hope that in the future, there would be found more conventional truths of these results. By more conventional, I mean of course purely algebraic proof. But I think this is visual thinking and therefore one has really to say that the contribution made by the model theories especially, are really very impressive for algebraes and I personally find these things very hard to understand.

As an algebraic also, one would certainly be interested in knowing how this algebraically-closed groups are really constructed and then one would hope for getting an information on other types of isomorphisms of these groups and also this is probably almost helpless since one cannot only prove that many finitely generated groups can be imbedded in these groups but also certain types of group which are not finitely generated can be imbedded always in a non-trivial algebraically-closed group. So, all these questions are really of high complexity.

I would like to mention the fact that the result which was mentioned by Fr. Nebres, that the automorphism group has become very big and would lead to the question, how the automorphism group module slowly became part of the automorphism group. The so-called automorphism which look like, I would be very interested in knowing that, but I think nobody knows this. Also, the connection to verbally complete groups I would like to know, because these verbally complete groups are very closely connected to the algebraically-closed groups, but of course we cannot go to all these technical details.

I just wanted to mention certain things which I have been doing in this country, since coming to this country 3 years ago, I have tried to get some people here in the country interested in the things I have been doing, especially certain questions in the theory of finite groups which is in comparison to this algebraically-closed group, a lot easier and I am quite happy that I found a number of people in the country who worked very well together with me and especially I would like to mention the on-going Ph.D. program in the country which seems to be successful.

I have been a visiting professor not only in the Philippines but also in other developing countries. I was in Brazil and I know their situation there relatively well. I have to say that the potential which is here in the country is bigger than in Brazil because I found that Filipinos are willing to accept tremendous effort in studying mathematics and I would really hope that these talented young people get all the support which is necessary for doing these things. I will probably stay a little bit longer in the country and I have more students who work together with me and I'm very glad at the outcome of these results, some of the results we got are impressive and I'm happy that I could be of help a little bit in this on-going effort and I would like to say finally, that I really hope that this program will go on and will get support from all the countries.

Jose Marasigan, Ph.D., Discussant

My comment would be of a more general nature. From the mathematical conferences I have attended. (I was at the International Congress of Mathematicians in Helsinki, and then in April this year, I was in Japan where I was fortunate enough to have been invited to attend the Spring meeting of the Japan Mathematical Society) I have been more and more convinced that algebra should be more developed in our country.

When you go to these conferences, you would notice, for example, that the language of algebra and geometry is presumed. If you want to be abreast of what's going on in these conferences, you need to know algebraic geometry. Also I would like to mention the fact that algebra is something that is not as useless as it has often been supposed to be. A very good example was the recent lecture of Dr. Eduardo Mendoza (University of Wuppertal) where he showed the applications of HADAMARD matrices in coding and information theory. Also when I was in Darmstadt a year ago, I found out that mathematical ideas considered before as purely abstract, are now being applied in areas like data structure, and data management.

In closing, I think there is a need in our country to develop algebra and I hope that the authority will give all the support we need in the development of algebra. We need the assistance of experts from other countries in our initial endeavor. From Japan, for example, we can invite many mathematicians. I have talked to some of them and they are willing to come and give lectures in algebraic geometry and other areas of mathematics.