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THE SIGN MATRIX CONCEPT AND SOME APPLICATIONS IN ABSTRACT ALGEBRA AND THEORETICAL PHYSICS

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ABSTRACT

This paper introduces the concept of the mxn **sign matrix** (or **Z-matrix**) $Z = (z_{ij})$, over the number set $F = \{+1, -1\}$, where $z_{ij} = \pm 1$ (or simply + or −) for every i=1,...,m and for every j=1,...,n (m, n any two positive integers). The *Hadamard matrix* is a special kind of nxn Z-matrix whose rows are mutually orthogonal. Given any two mxn Z-matrices $Z_a = ([z_a]_{ij})$ and $Z_b = ([z_b]_{ij})$, we define their **star product**, $Z_a \star Z_b$, to be the matrix $Z_c = ([z_c]_{ij})$, where $[z_c]_{ij} = [z_a]_{ij} \cdot [z_b]_{ij}$ for all i=1,...,m, j=1,...,n and · is ordinary multiplication of real numbers. Under this matrix operation, ★, the set Z(mxn) of all the 2^{mxn} possible mxn sign matrices form an abelian *p-group* of order 2^{mxn} isomorphic to the **Klein group** of the same order. Z-matrices can be used to construct a family of division algebras of order 2^r (r any positive integer) over the real numbers as well as special groups (such as the group of **Dirac operators** in quantum electrodynamics) and pseudogroups with important applications in pure mathematics and theoretical physics.

Introduction

The positive (+) and negative (−) signs feature in a wide variety of disparate disciplines such as mathematics, philosophy, science, and art. They are ubiquitous as symbols of bipolarities as mundane as life and death, love and hate, debit and credit, and as esoteric as thesis and antithesis, matter and antimatter, Yin and Yang, etc. These two symbols represent primitive entities without which mathematics as we know it today will not exist. In fact, the simplest nontrivial mathematical system consists of just these two entities; it is isomorphic to the smallest group.

In this paper, some of the interesting and important properties of these primitive entities will be deduced by introducing the concept of the *sign matrix*

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(or *Z-matrix*). This is a special kind of matrix whose entries are the sign symbols + and − representing the numbers +1 and −1. We shall prove that these Z-matrices form groups of order 2^r (r any positive integer) which have the abstract structure of the *Klein group* (elementary p-group) of order 2^r. Moreover, we shall also show that Z-matrices can be used to construct division algebras of order 2^r over the real numbers as well as special pseudogroups and groups (such as the group of *Dirac operators* in quantum electrodynamics and the group of rotations in six dimensions) with important applications in pure mathematics and theoretical physics.

§ –2 The Sign Matrix Z

Let us define a special matrix which we shall call the *sign matrix* or *Z-matrix* all of whose entries are elements of the number set $F = \{+1, -1\}$ or simply $\{+, -\}$.

DEFINITION 1. A *sign matrix* is an mxn matrix $Z = (z_{ij})$, where $z_{ij} = +1$ or -1 (or simply + or −), for every i=1,. . . . n and for every j=1,. . . ,n.

The simplest kind of sign matrix is the 1x1 with only one sign symbol as an entry. The 1xn Z-matrix is called a *row*; the mx1 or *column* Z-matrix is the transpose of the 1xn or row. Any mxn Z-matrix can be formed easily out of m 1xn Z-matrices; these 1xn matrices form the m rows of the resulting mxn Z-matrix. In particular, an nxn or *square* Z-matrix can be formed out of n 1xn Z-matrices.

THEOREM 1. (Let $Z = (z_{ij})$ be an mxn sign matrix, where i=1,. . . , m and j=1,. . . ,n. Then there are exactly $N(mxn) = 2^{mxn}$ possible mxn sign matrices all of which are distinct.

PROOF. We shall prove this theorem by forming any given mxn Z-matrix out of m 1xn Z-matrices. First, we determine the total number N(1xn) of all possible 1xn Z-matrices $Z = (z_{ij})$, where i=1 and j=1,. . . ,n. Since z_{ij} = + or −. This problem is equivalent to determining the number of linear arrangements of n sign symbols of at most two kinds, + and −, taken n at a time, where there are r of the kind − and (n-r) of the kind +, such that r = 0,1,2,. . . ,n. This can be shown to be:

$$N(1xn) = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} = 2^{1xn}$$

Since the mxn Z-matrix is formed out of m 1xn Z-matrices, then for each of the 2^n distinct 1xn Z-matrices chosen for the first row, there are exactly 2^n choices for the second row, or a total of $2^n x 2^n = 2^{2xn}$ distinct 2xn Z-matrices;

$2^n x 2^n x^{2n} = 2^{9xn}$ distinct 3xn Z-matrices;..., etc. Therefore, it follows that there are exactly a total of

$$N(mxn) = 2^n x 2^n x 2^n_x \ldots x 2^n = 2^{mxn} \qquad \text{Eq. (1)}$$

possible mxn Z-matrices all of which are distinct. Q.E.D.

It follows easily from Eq. (1) that if m = n = 1, then we have exactly $N(1x1) = 2^{1x1} = 2$ 1x1 Z-matrices: $Z_+ = (+)$ and $Z_- = (-)$. In general, if m = n, there are exactly $N(nxn) = 2^{nxn}$ square or nxn Z-matrices.

Let us illustrate the application of Theorem 1 by some examples. Thus, let us form all of the $N(1x2) = 2^{1x2} = 4$ possible 1x2 Z-matrices of the set Z(1x2). Since there are only n = 2 entries in each 1x2 Z-matrix, then the only possible linear arrangements of the signs + and – are the following:

Z(1x2): [+ +], [+ –], [– +], [– –]

These four 1x2 Z-matrices are all distinct and there are no other 1x2 Z-matrices than these. Using these four 1x2 Z-matrices, we can now form all of the N(2x2) $= 2^{2x2} = 16$ possible 2x2 Z-matrices.

```
Z(2x2):   + +    + +    + +    + +    + –    + –    + –    + –
          + +    + –    + –    – –    + +    + –    – +    – –

          – +    – +    – +    – +    – –    – –    – –    – –
          + +    + –    – +    – –    + +    + –    – +    – –
```

Again, these 16 square or 2x2 Z-matrices are all distinct; there are no other 2x2 Z-matrices than these. In the same way, we can form all of the N(3x3) = 512 possible 3x3 Z-matrices out of the N(1x3) = 8 possible 1x3 Z-matrices,..., etc.

§ 3 – The Star Product of Two Matrices

For sign matrices to be useful, we must introduce a rule of composition for them. Although Z-matrices can be subjected to the standard matrix operations of addition, multiplication by a scalar, and ordinary matrix multiplication, not much of interest or importance can be gained from them by this means. Standard matrix theory evolved from the algebra of linear transformations and most of the common matrix operations have been defined to reflect the properties of such transformations. If we are to deal with mathematical concepts other than linear transformations, it becomes necessary to introduce other kinds of matrix operation which, among other things, will have useful applications in such fields as group theory and related matters as well as in theoretical physics and other branches of applied mathematics. In particular, we seek for an operation * which can combine Z-matrices in a fruitful way.

DEFINITION 2. Let $A = (a_{ij}) \cdot$ and $B = (b_{ij})$ be any two mxn matrices over a field F. The *star product*, $A * B$, of A and B is the mxn matrix $C = (c_{ij})$, where

$$c_{ij} = a_{ij} \cdot b_{ij}$$

for every $i=1,\ldots,m$ and for every $j=1,\ldots,n$, and the operation $\cdot$ is multiplication in F.

The operation * shall be called *star multiplication* to distinguish it from ordinary matrix multiplication.

It is clear from Definition 2 that the star product can be applied to any two matrices of the same dimensions mxn with entries from any given field or set F with a well defined operation, $\cdot$, of multiplication. Thus, if $Z_a = ([Z_a]_{ij})$ and $Z_b = ([z_b]_{ij})$ are any two mxn Z-matrices over the number set $F = \{+1, -1\}$, then the *star product*, $Z_a * Z_b$, of Z_a and Z_b is the mxn Z-matrix $Z_c = ([z_c]_{ij})$, where

$$[z_c]_{ij} = [z_a]_{ij} \cdot [z_b]_{ij}$$

for every $i=1,\ldots,m$ and for every $j=i,\ldots\ldots,n$. Since the elements of the set F are the numbers +1 and −1, then they satisfy the following composition rule:

$$(+1)\cdot(+1) = (-1)\cdot(-1) = +1$$
$$(+1)\cdot(-1) = (-1)\cdot(+1) = -1$$

This rule shows that the number set $F = \{+1, -1\}$ is closed under the operation $\cdot$ of multiplication; they form a group $<F; \cdot>$ isomorphic to the cyclic group C_2 of order 2. In this system, +1 is the identity element of F and is thus of order 1. On the other hand, the element −1 is of order 2. Because of this composition rule, it is easy to see that the star product of any two mxn Z-matrices is always an mxn Z-matrix. This indicates that the operation * is a closed binary operation over certain sets of Z-matrices.

§ 4 – Sign Matrix Groups

We shall now prove that the Z-matrices satisfy the postulates of a group under the operation * of star multiplication given by Definition 2.1.

THEOREM 2. The set Z(mxn) of all mxn Z-matrices and the operation * of star multiplication form a commutative group $<Z; *>$ of order 2^{mxn}.

PROOF. Let Z = Z(mxn) be the set of all the 2^{mxn} possible and distinct mxn Z-matrices. It follows trivially from Def. 2.1 that the star product of any two mxn Z-matrices is also an mxn Z-matrix. Since Z(mxn) contains all possible mxn Z-matrices, then operation * is *closed* over Z(mxn). We assert that the system

<Z; *> is a commutative group of order 2^{mxn}, where m and n are any two positive integers. To prove this, it is sufficient to show that (a) the mxn Z-matrices satisfy the *associative* postulate under *, (b) there is a unique mxn Z-matrix Z_1 with the properties of a unique *identity element* under *, (c) every mxn Z-matrix Z_u has a unique *inverse* Z^{-1}_u in Z(mxn), and (d) the system <Z;*> is *commutative*.

(a) Let $Z_a = ([z_a]_{ij})$, $Z_b = ([z_b]_{ij})$, and $Z_c = ([z_c]_{ij})$ be any three mxn Z-matrices. Using Def. 2.1, form the triple products: $Z_a*(Z_b*Z_c) = Z_x$, $(Z_a*Z_b)*Z_c = Z_y$, where $Z_x = ([z_x]_{ij})$, $Z_y = ([z_y]_{ij})$, and

$$[z_x]_{ij} = [z_a]_{ij} \cdot ([z_b]_{ij} \cdot [z_c]_{ij}), \; [z_y]_{ij} = ([z_a]_{ij} \cdot [z_b]_{ij}) \cdot [z_c]_{ij}$$

Since <F; •> is a group and the entries of any mxn Z-matrix are the numbers $+1, -1 \in F$, then they always satisfy the associative postulate under multiplication, •, in F. This implies that: $Z_a*(Z_b*Z_c) = (Z_a*Z_b)*Z_c$ for all Z_a, Z_b, and Z_c in the set Z(mxn). Therefore, the system {Z;*} is *associative*.

(b) Let $Z_1 = ([z_1]_{ij})$, where $[z_1]_{ij} = +1$ for every i=1,...,m and for every j=1,...,n, and +1 is the identity element of *F*. Then for any mxn Z-matrix Z_u in Z(mxn), it follows from Def. 2.1 that: $Z_1 * Z_u = Z_u * Z_1 = Z_u$. Clearly, Z_1 is unique since any other $\underline{Z}_1$ with the same properties can be shown to be such $\underline{Z}_1 = Z_1$. Therefore Z_1 is a unique *identity element* under *. We shall also call Z_1 the mxn *unit Z-matrix*.

(c) Let $Z_a = ([z_a]_{ij})$ be any mxn Z-matrix and let $Z^{-1}_a = ([z'_a]_{ij})$ be some mxn Z-matrix such that: $Z^{-1}_a * Z_a = Z_a*Z^{-1}_a = Z_1$. For this equation to be true, we must always have:

$$[z'_a]_{ij} \cdot [z_a]_{ij} = [z'_a]_{ij} \cdot [z'_a]_{ij} = +1$$

for every i=1,...,m and for every j=1,...,n. Since $[z'_a]_{ij} = \pm 1$ and $[z_a]_{ij} = \pm 1$, then by the composition rule of <F; •> this equation can be true if and only if $[z'_a]_{ij} = [z_a]_{ij}$ for all values of i and j. This implies that $Z^{-1}_a = Z_a$ which means that every Z-matrix Z_a is *self-inverse*, that is, $Z_a*Z_a = Z_1$ for all Z_a in Z(mxn). Moreover, it is also clear that Z_a can not have any other inverse than itself so that Z^{-1}_a is unique. This result is also obvious from the fact that every element of F is self-inverse.

(d) Let $Z_a = ([z_a]_{ij})$ and $Z_b = ([z_b]_{ij}$ be any two mxn Z-matrices. Form the star products: $Z_a * Z_b = Z_x$, $Z_b * Z_a = Z_y$, where $Z_x = ([z_x]_{ij})$, $Z_y = ([z_y]_{ij}$,

$$[z_b]_{ij} = [z_a]_{ij} \cdot [z_b]_{ij} \text{ and } [z_y]_{ij} = [z_b]_{ij} \cdot [z_a]_{ij}$$

Since $[z_a]_{ij}$ and $[z_b]_{ij}$ are the numbers $+1, -1 \in F$, then they satisfy the commutative postulate under multiplication, •, in F. Thus, $[z_a]_{ij} \cdot [z_b]_{ij} = [z_b]_{ij} \cdot [z_a]_{ij}$ is always true for every i=1,...,m and for every j=1,...,n. This implies that $Z_a * Z_b = Z_b * Z_a$ for all mxn Z-matrices Z_a and Z_b. Therefore, <Z; *> is commutative. **Q.E.D.**

The above arguments, (a) to (d), all follow from Def. 2.1 and the fact that <F; •> is a group; every basic group property of <Z;*> is derived from <F; •>. Thus, * is associative and commutative because • has these properties, etc.

THEOREM 3. Every Z-matrix group <Z, *> of order 2^r is isomorphic to the *Klein group* <K_s; o> of the same order, where r is any positive integer.

PROOF. In Theorem 2 we proved that the set Z(mxn) of all mxn Z-matrices forms a commutative group <Z; *> of order 2^{mxn} under star multiplication *. This group is such that every $Z_u \in Z$, except the identity element Z_1, is of order 2. Each of these order 2 elements generates a subgroup of order 2 which is isomorphic to the cyclic 2-group <C_2; o>. Now, let Z_a and Z_b be two distinct elements of order 2 of Z and let <A; *> and <B; *> be the subgroups of order 2 generated by Z_a and Z_b, respectively. Then $A \cap B = \{Z_1\}$. Form the set

$$AB = \{Z_x | Z_x = Z_{\underline{a}} * Z_{\underline{b}}, Z_{\underline{a}} \in A, Z_{\underline{b}} \in B\}$$

where $Z_{\underline{a}}$ is any element of A and $Z_{\underline{b}}$ is any element of B. It can be easily shown that AB has exactly 2x2 = • 4 distinct elements and that the system <AB; *> is a group of order 4 which is isomorphic to the *direct product*, AXB, of A and B. Similarly, let Z_c be an element of order 2 from the subset Z-AB of Z and let <C;*> be the subgroup of order 2 generated by Z_c. Then it is clear that $AB \cap C = \{Z_1\}$. Again, form the set

$$ABC = \{Z_y | Z_y = Z_{\underline{a}} * Z_{\underline{b}}, * Z_{\underline{c}}, Z_{\underline{a}} \in A, Z_{\underline{b}} \in B, Z_{\underline{c}} \in C\}.$$

Then we can again show that <ABC; *> is a group of order 2x2x2 = 8 which is isomorphic to the direct product AXBXC. By extending the same argument to the remaining elements of order 2 in the subset Z-ABC, etc., we finally exhaust all of the elements of Z and obtain the set ABC...R:

$$ABC\ldots R = \{Z_w | Z_w = Z_{\underline{a}} *,,, * Z_{\underline{r}}, Z_{\underline{a}} \in A, \ldots, Z_{\underline{r}} \in R\}.$$

of order 2^r which contains all of the 2^{mxn}, elements of Z = Z(mxn). This simply means that

$$Z = ABC\ldots R \text{ (to r subgroups)}$$

and therefore it follows that

$$Z \cong A \times B \times C \times \ldots \times R \text{ (to r subgroups).}$$

Since each of the subgroups A,B,C,...,R is a isomorphic to the cyclic group C_2 of order 2 and Z is of order 2^{mxn}, then it follows that r = mxn and that we finally have:

$$Z \cong K_s = C_2 X C_2 X C_2 X \ldots X C_2 \text{ (to r=mxn subgroups)}$$

We shall call the elementary p-group $<K_s; \circ>$ the *Klein group of degree r* and order $s = 2^r$. This group is commutative and all of its elements are self-inverse. This completes the proof of the theorem. **Q.E.D.**

§ 5 – Z-Representation of Klein Groups

In the foregoing sections, we have shown that it is always possible to form a set Z(mxn) of 2^{mxn} Z-matrices, where m and n are any two positive integers. Moreover, we have proved that the system $<Z; *>$ of order 2^{mxn}, is always a group isomorphic to the Klein group of the same order. Therefore, we have the following:

> THEOREM 3.1. Every Klein group $<K_s; \circ>$ of order $s = 2^r$ (r any positive integer) can be *represented* by a Z-matrix group $<Z; *>$ or order 2^r and dimensions mxn, where $mxn \geq r$.

PROOF. By Theorem 3, every Z-matrix group of order 2^r is isomorphic to the Klein group of the same order. Therefore, every Klein group of order $s = 2^r$ is isomorphic to some Z-matrix group of the same order. For reference, let us call the dimensions mxn of the Z-matrices in Z = (mxn) the *dimensions* of the Z-matrix group {Z; *}. To prove this theorem, we must therefore show that $mxn \geq r$. If $<K_s; \circ>$ is of order 2^r, then its Z-matrix representation must be of the same order regardless of its dimensions mxn. This means that the smallest representation of $<K_s; \circ>$ is of dimensions mxn such that mxn = r. Thus, if r is a prime, then we can have m = 1, n = r, or m = r, n = 1; if r is composite with prime factors $r_1, r^2, \ldots, r_1$, then m and n can be any combination of these prime factors that satisfy the condition mxn = r. Next, consider the case when mxn > r. Are there Z-matrix groups of order 2^r and of dimensions mxn > r? The order of any Z-matrix group is of the form p^k, where p = 2 and k is a positive integer. Since 2 is a prime, then this group of order 2^k has a series of proper subgroups of orders $2^{k-1}, 2^{k-2}, \ldots, 2^r, \ldots 2^2, 2^1$, all of which are Z-matrix groups. This implies that every Z-matrix group of order 2^r is isomorphic to a subgroup of a Z-matrix group of order 2^k, where k > r. Now, let this Z-matrix group of order 2^k be of dimensions mxn. Therefore, any subgroup of order 2^r of this group of order 2^k is also of dimensions mxn. But k > r and k = mxn; hence $mxn \geq r$. **Q.E.D.**

As an illustration, let us find four Z-matrix representations of the Klein group $<K_s; \circ>$ of order $2^2 = 4$ and dimensions $mxn \geq 2$. If mxn = 2, then we can have m = 1, n = 2, that is mxn = 1x2. And if mxn > 2, then there are many possibilities: mxn = 2x2, 1x4, 4x4, etc. Thus, let us take the sets Z, A, B, and C given in Figure 1. These sets from Z-matrix groups of order $2^2 = 4$, viz. $<Z; *>$, $<A; *>$, $<B; *>$ and $<C; *>$ all of which are isomorphic to each other and to the Klein group $<K_s; \circ>$ of the same order.

The Klein group <K_s; o> (or simply K_s) is a very interesting system with a simple and beautiful structure. It is a commutative p-group whose subgroups are all Klein groups; K_2 is regarded as the basic K_s-group. The most widely known Klein group is K_4 which is popularly known as the *Klein four group.* This group, among other things, is used to describe the symmetries of the rectangle as well as certain symmetry classes in the relativistic theory of particle spin. Other Klein groups are involved in division algebras, in Dirac's electron theory, and in other fields of pure and applied mathematics.

$Z_1 = [+\,+]$ $\quad Z_2 = [+\,-]$ $\quad A_1 = \begin{matrix} + & + \\ + & + \end{matrix}$ $\quad A_2 = \begin{matrix} + & + \\ + & - \end{matrix}$

$Z_9 = [-\,+]$ $\quad Z_4 = [-\,-]$ $\quad A_9 = \begin{matrix} + & + \\ - & + \end{matrix}$ $\quad A_4 = \begin{matrix} + & + \\ - & - \end{matrix}$

$Z = (\underline{Z}(1x2)$ $\qquad A = \underline{Z}(2x2)$

$B_1 = [++++]$ $\quad B_2 = [+--+]$ $\quad C_1 = \begin{matrix} ++++ \\ ++++ \\ ++++ \\ ++++ \end{matrix}$ $\quad C_2 = \begin{matrix} +--+ \\ +--+ \\ +--+ \\ +--+ \end{matrix}$

$B_3 = [+-+-]$ $\quad B_4 = [++--]$ $\quad C_3 = \begin{matrix} +-+- \\ +-+- \\ +-+- \\ +-+- \end{matrix}$ $\quad C_4 = \begin{matrix} ++-- \\ ++-- \\ ++-- \\ ++-- \end{matrix}$

$B = \underline{Z}(1x4)$ $\qquad C = \underline{Z}(4x4)$

	Z_1	Z_2	Z_3	Z_4
Z_1	Z_1	Z_2	Z_3	Z_4
Z_2	Z_2	Z_1	Z_4	Z_3
Z_3	Z_3	Z_4	Z_1	Z_2
Z_4	Z_4	Z_3	Z_2	Z_1

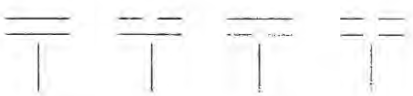
<Z; *>

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

<K_4; o>

FIGURE 1. Four Z-matrix representations of the Klein group <K_4; o> of order $2^2 = 4$. The Z-matrix groups <Z; *>, <A; *> <B; *>, and <C; *> are of dimensions 1x2, 2x2, 1x4, and 4x4, respectively, but they are all of the same order $2^2 = 4$. Moreover, they are all isomorphic to each other and to <K_4; o>. The sets A = $\underline{Z}$ (2x2), B = $\underline{Z}$(1x4), and C = $\underline{Z}$ (4x4) are subsets of order 4 of the Z-matrix sets Z (2x2), Z (1x4), Z(4x4), respectively.

It is interesting to note that the Chinese YIN-YANG anagrams in the *I Ching (Book of Changes* – the first book of the *Confuscian Classics*) can be represented by Z-matrices. Let the Yang line _____ and the Yin line — — be represented by + and –, respectively. Then the four basic *bigram* configurations become:



bigrams

+	–	+	–	
+	+	–	–	2x1 Z-matrices
Z_1	Z_2	Z_3	Z_4	

Similarly, the eight *trigrams* are:

trigrams

+	–	+	–	+	–	+	+	3x1 Z-matrices
+	+	–	–	+	+	–	–	
+	+	+	+	–	–	–	–	
Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	

We see from these that the anagrams can be formed in exactly the same way as the mx1 Z-matrices, where m is the number of rows of Z. There are therefore exactly $N(mx1) = 2^m$ anagrams, m=2,3,6. Thus, if m = 6, there are exactly $2^6 = 64$ *hexagrams*, eight of which (in Z-matrix from) are shown below:

+	–	+	–	+	–	+	–
+	+	–	–	+	+	–	–
+	+	+	+	–	–	–	–
+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+

Since each set of $N(mx1) = 2^m$, m=2,3,6, anagrams can be represented by mx1 Z-matrices, then they form groups of order 2^m isomorphic to the corresponding Klein groups of the same order. Thus, the set of $N(6x1) = 2^6 = 64$ *hexagrams* form a group isomorphic to the Klein group of order 64. This Klein group contains subgroups of orders $2^5 = 32$, $2^4 = 16$, $2^3 = 8$, $2^2 = 4$, and $2^1 = 2$.

The subgroups of orders 4 and 8 are isomorphic to the groups of *bigrams* and *trigrams*, respectively.

§ 6 – Division Algebra Over the Real Numbers

Let us now consider an important application of Z-matrices in abstract algebra. For this and other applications we need to introduce an interesting and useful matrix called the $\otimes$*-multiplication matrix* of a set K_s.

DEFINITION 3. Let $<K_s; \circ>$ be a Klein group of order $s = 2^r$, where $K_s = \{e_i | i=1, \ldots, s = 2^r\}$; $S_r(K_s) = (e_{ij})$ the defining structure matrix of $<K_s; \circ>$, where $e_{ij} = e_i \circ e_j$ for all $i,j=1,\ldots,s$; $Z_r(K_s) = (z_{ij})$ a given Z-matrix, where $Z_{ij} = \pm 1$ for all $i,j=1,\ldots,s$. The sxs matrix

$$M_r(K_s) = Z_r(k_s) .* S_r(K_s) = (m_{ij})$$

is called $\otimes$*-multiplication matrix* of k_s, where

$$m_{ij} = e_i \otimes e_j = z_{ij} \cdot e_{ij} = z_{ij} \cdot (e_i \circ e_j)$$

for all $i, j=1,\ldots s$.

The $\otimes$-multiplication matrix defined above determines the nature of the operation $\otimes$ over the set K_s. Such a matrix can be used to construct division algebras and special groups and pseudogroups; starting with a Klein group, new systems are formed by means of Z-matrices. The Klein group is thus the *substratum* of such systems.

Consider the algebra $A_r = \{\gamma, F; +, x, \otimes, \otimes,.\}$ of order $s = 2^r$ over the field F. Take as the *basis* of the vector space γ the set $K_s = \{e_i \mid i=1,\ldots,s\}$ of s *basis vectors* over which the binary operation $\otimes$ is defined by the $\otimes$-multiplication matrix $M_r(K_s), = (m_{ij})$, where $m_{ij} = e_i \otimes e_j$ for all $i,j=1,\ldots,s$. Every vector of this algebra A_r can be expressed uniquely as a linear combination of the s basis vectors in K_s. Thus, if **a**, **b** E A_r, then

$$a = \sum_{i=1}^{s} a_i \cdot e_i \quad \text{and} \quad b = \sum_{j=1}^{s} b_j \cdot e_j,$$

where $a_j, b_j \in F$. *Vector multiplication* is defined by *bilinearly* and the matrix $M_r(K_s)$ so that the product, **a** () **b**, of any two vectors **a**, **b** $\in A_r$ is given by the expression.

$$a \otimes b = \sum_{ij=k} f_{ij} z_{ij} \cdot e_k \qquad (k=1,\ldots,s) \qquad \text{Eq. (1)}$$

$i,j=1,\ldots,s$, where $f_{ij} = a_i b_j$, e_k E K_s, and the sum is to be extended over all pairs of indices ij for which $e_i \otimes e_j = z_{ij} \cdot e_k$.

By definition, an algebra A_r over a field F is a *division algebra* if it has a *unity* e_1 of vector multiplication and every non-zero vector $a \in A_r$ has a unique *inverse* $a^{-1} \in A_r$, that is, a vector with the property that $a \otimes a^{-1} = e_1$. Such a vector a^{-1} exist in A_r if a vector a^*, called the *conjugate* of a, exists in A_r, with the special property that

$$a \otimes b^* = \sum_{ij=k} f_{ij}\, z_{ij} \cdot e_k \quad = N(a) \cdot e_1, \qquad \text{Eq. (1.1).}$$

where $N(a) > 0$, called the *norm* of **a**, is an element of the field F and $f_{ij} = a_i a^*_j$. This implies that all the terms of the expression $\sum_{ij=k} f_{ij}\, z_{ij} \cdot e_k$ for which $k \neq 1$ all *add up to zero;* only the terms where $k = 1$ have a *non-zero sum*, that is,

$$\sum_{ij=k} f_{ij}\, z_{ij} \cdot e_k \quad = \sum_{ij=1} f_{ij}\, z_{ij} \cdot e_1 = N(a) \cdot e_1 \qquad \text{Eq. (1.2)}$$

Therefore, $N(a) = \sum_{ij=k} f_{ij}\, z_{ij}$ (summed over all index pairs ij for which $e_i \odot e_j = z_{ij} \cdot e_1$). With such a vector $a^* \in A_r$, we find that the inverse a^{-1} of $a \neq 0$ exists in A_r and is given by

$$a^{-1} = a^*/N(a). \qquad \text{Eq. (2)}$$

The problem of constructing a division algebra over the real numbers is thus equivalent to the problem of constructing a $\otimes$-multiplication matrix $M_r(K_s)$ such that the conjugate a^* of every vector $a \neq 0$ can be defined that satisfies the requirements of Eq. (1.1). Such a matrix can indeed be constructed by means of Definition 3 and by defining the conjugate of any non-zero vector $a = \sum_{i=1}^{s} a_i e_i = a_1e_1+a_2e_2 +\ldots+ a_se_s$ to be

$$a^* = a_1e_1 - (a_2e_2 +\ldots+ a_se_s), \qquad \text{Eq. (2.1)}$$

where $a_1 \in F$ and $e_i \in K_s$. All known division algebras over the real numbers *(complex numbers, quaternions, Cayley numbers)* satisfy the above requirements. Moreover, in these algebras, the norm of any vector $a \neq 0$ is a positive real number given by the expression

$$N(a) = \sum_{i=1}^{s} a_i^2 = a_1^2 + \ldots + a_s^2 \qquad \text{Eq. (2.2)}$$

To construct the required $\otimes$-multiplication matrix $M_r(K_s) = (m_{ij})$ for a division algebra A_r over F of order $s = 2^r$, we first form the matrices

$Z_r(K_s) = (z_{ij})$ and $S_r(K_s) = (e_{ij})$, where $z_{ij} = \pm 1$ and $e_{ij} = e_i \circ e_j$ for all $i, j=1,\ldots,s$. To do this, we note that the basis vectors of the algebras of *complex numbers* $(r = 1)$, *quaternions* $(r = 2)$, and *Cayley numbers* $(r = 3)$, all satisfy the following equations:

$$
\begin{array}{rcll}
e_i \otimes e_j & = & z_{ij} \cdot (e_i \circ e_j) & \text{(for all } i, j=1,\ldots,s) \\
e_i \otimes e_j & = & e_j \otimes e_i & (i \neq j, j \neq 1) \\
e_i \otimes e_1 & = & e_1 \otimes e_j = e_i & \text{(for all } i=1,\ldots,s)
\end{array}
\qquad \text{Eqs. (3)}
$$

where $e_i, e_j \in K_s$, $s = 2^r$, and $z_{ij} = \pm 1$. This shows that the matrix $Z_r(K_s) = (z_{ij})$ is such that

$$
\begin{array}{rcll}
z_{ij} & = & \begin{array}{ll} 1 \text{ if } i-j = \text{even} & (i \geq j,\ i, j \neq 1) \\ 1 \text{ if } i-j = \text{odd or } 0 & (i \geq j,\ i, j \neq 1) \end{array} & \\
z_{ij} & = & -z_{ji} & (i \neq j,\ i, j \neq 1) \\
z_{i1} & = & -z_{1i} = z_{11} = +1 & \text{(for all } i=1,\ldots,s)
\end{array}
\qquad \text{Eqs. (3.1)}
$$

To illustrate this, we show in Figure 2 the matrix $S_r(K_s)$ and a matrix $Z_r(K_s)$ satisfying Eqs. (3.1) which can be used to construct a matrix $M_r(K_s) = Z_r(K_s) * S_r(K_s) = (m_{ij})$ satisfying Eqs. (3). This matrix $M_r(K_s)$ is shown in simplified form in Figure 3 where we have set $\pm = \pm 1$ and $v = e_v$. Also, note that we have indicated some submatrices of $Z_r(K_s)$, $S_r(K_s)$, and $M_r(K_s)$.

It can be shown that this matrix $M_r(K_s)$, where r is any positive integer, defines an operation $\otimes$ over the set K_s which can be used as the basis of an algebra A_r of order $s = 2^r$ over the field F of real numbers. It is easy to verify that if we use Eq. (2.1) to define the conjugate, **a***, of any non-zero vector $\mathbf{a} \in A_r$, then a has a norm $N(a) = a_1^2 + \ldots + a_s^2$ as given by Eq. (2.2) and a unique inverse $a^{-1} = a^*/N(a)$ as given by Eq. (2). Thus, A_r is a division algebra over F.

$$
\begin{array}{c|cccccc}
 & + & + & + & + & + & +\ldots+ \\
Z_1 & + & - & + & - & + & +\ldots- \\
 & + & - & - & + & - & +\ldots+ \\
Z_2 & + & + & - & - & + & -\ldots- \\
 & + & - & + & - & - & +\ldots+ \\
 & + & + & - & + & - & -\ldots- \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & + & - & + & - & + & -\ldots-
\end{array}
\qquad
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6\ldots n_o \\
S_1 & 2 & 1 & 4 & 3 & 6 & 5\ldots n_1 \\
 & 3 & 4 & 1 & 2 & 7 & 8\ldots n_2 \\
S_2 & 4 & 3 & 2 & 1 & 8 & 7\ldots n_3 \\
 & 5 & 6 & 7 & 8 & 1 & 2\ldots n_4 \\
 & 6 & 5 & 8 & 7 & 2 & 1\ldots n_5 \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\
 & n_o & n_1 & n_2 & n_3 & n_4 & n_5\ldots 1
\end{array}
$$

(a) $Z_r(K_s)$ (b) $S_r(K_s)$

FIGURE 2. (a) $Z_r(K_s) = (z_{ij})$ is a special sxs Z-matrix, (b) $S_r(K_s) = (e_{ij})$ is the sxs structure matrix of the Klein group $<K_s; \circ>$ of order $s = 2^r$; $n = 2^r\text{-}k$, k=0,1, . . . , $(2^r\text{-}1)$; $\pm = \pm 1$ and $v = e_v$.

A simple examination of the entries of the three matrices $Z_r(K_s)$, $S_r(K_s)$, and $M_r(K_s)$ will show that they can be partitioned into unique submatrices Z_1, Z_2, Z_3, Z_4, etc.; S_1, S_2, S_3, S_4, etc.; M_1, M_2, M_3, M_4, etc.; where Z_u, S_u, and M_u are of dimensions vxv such that $v = 2^u$ and $u \leq r$. These submatrices (indicated by dotted lines in Figures 2 and 3) are elements of the following ascending series:

$$Z_1 < Z_2 < Z_3 < Z_4 \ldots < Z_u < \ldots < Z_r$$
$$S_1 < S_2 < S_3 < S_4 \ldots < S_u < \ldots < S_r$$
$$M_1 < M_2 < M_3 < M_4 \ldots < M_u < \ldots < M_r$$

Clearly, each submatrix Z_u, S_u, or M_u of Z_r, S_r, or M_r, respectively, can be treated on its own as a Z, S, or M matrix of smaller dimensions.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$..n_0$
M_1	2	-1	4	-3	6	-5	8	-7	10	-9	12	-11	14	-13	16	-15	$..-n_1$
	3	-4	-1	2	-7	8	-5	6	-11	12	-9	10	-15	16	-13	14	$..n_2$
M_2	4	3	-2	-1	8	-7	6	-5	12	-11	10	-9	16	-15	14	-13	$..-n_3$
	5	-6	7	-8	-1	2	-3	4	-13	14	-15	16	-9	10	-11	12	$..n_4$
	6	5	-8	7	-2	-1	4	-3	14	-13	16	-15	10	-9	12	-11	$..-n_5$
	7	-8	5	-6	3	-4	-1	2	-15	16	-13	14	-11	12	-9	10	$..n_6$
M_3	8	7	-6	5	-4	3	-2	-1	16	-15	14	-13	12	-11	10	-9	$..-n_7$
	9	-10	11	-12	13	-14	15	-16	-1	2	-3	4	-5	6	-7	8	$..n_8$
	10	9	-12	11	-14	13	-16	15	-2	-1	4	-3	6	-5	8	-7	$..-n_9$
	11	-12	9	-10	15	-16	13	-14	3	-4	-1	2	-7	8	-5	6	$..n_{10}$
	12	11	-10	9	-16	15	-14	13	-4	3	-2	-1	8	-7	6	-5	$..-n_{11}$
	13	-14	15	-16	9	-10	11	-12	5	-6	7	-8	-1	2	-3	4	$..n_{12}$
	14	13	-16	15	-10	9	-12	11	-6	5	-8	7	-2	-1	4	-3	$..-n_{13}$
	15	-16	13	-14	11	-12	9	-10	-7	-8	5	-6	3	-4	-1	2	$..n_{14}$
M_4	16	15	-14	13	-12	11	-10	9	-8	7	-6	5	-4	3	-2	-1	$..-n_{15}$
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	...
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	...
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	...
	n_0	$-n_1$	n_2	$-n_3$	n_4	$-n_5$	n_6	$-n_7$	n_8	$-n_9$	n_{10}	$-n_{11}$	n_{12}	$-n_{13}$	n_{14}	$-n_{15}$	$..-1$

FIGURE 3. General form of ⊗-multiplication matrix $M_r(K_s) = Z_r(K_s) * S_r(K_s) = (m_{ij})$, where $m_{ij} = e_i \otimes e_j = z_{ij} \cdot (e_i \circ e_j)$; $n_k = 2^r\text{-}k$, $k = 0,1, \ldots, (2^r\text{-}1)$ and $v = e_v$. This matrix defines a family of *Cayley Algebras* of degree r!

Because the matrix M_r contains all other smaller matrices M_u, where $u < r$, as submatrices, then every algebra A_r also contains as subalgebras all other smaller algebras A_u, $u < r$, of the same type. Thus, we also have the following ascending series:

$$A_o < A_1 < A_2 < A_3 < A_4 \ldots < A_u < \ldots < A_r$$

where we have included $A_o = F$ for completeness. This shows that all A_r algebras defined by the matrix M_r (K_s) have a common underlying structure. If $r \geq 3$, then A_r is non-associative; only A_o (real numbers), A_1 (complex numbers), and A_2 (quaternions) are associative.

We note, however, that Eqs. (3.1) can be satisfied by many other Z-matrices and that the matrix Z_r (K_s) shown in Figure 2 is therefore not unique. Each matrix Z_r (K_s) satisfying Eqs. (3.1) determines a $\otimes$-multiplication matrix M_r (K_s) satisfying Eqs. (3.) The set of all such matrices therefore determines a class of division algebras A_r of degree r, where r is any positive integer. The members of the class of algebras A_3 of order 8 are *alternative algebras* known as *Cayley-Dickson algebras.*

Figure 4 shows the $\otimes$-multiplication matrix $M_3(K_8)$ that defines the algebra U_3 which is isomorphic to the algebra of *Cayley numbers* (order 2^3=8). $M_3(K_8)$ can be seen to contain the submatrices $M_2(K_4)$ and $M_1(K_2)$ which define the algebras U_2, (order 2^2=4) and U_1 (order 2^1=2), respectively. These algebras, in turn, can be shown to be isomorphic to the algebras of quaternions and complex numbers, respectively. This shows that the algebra of Cayley numbers contains the quaternions and complex numbers as subalgebras. Any algebra U_r, where $r \geq 3$, is *non-associative*. The only

	1	2	3	4	5	6	7	8
M_1	2	-1	4	-3	6	-5	-8	7
	3	-4	-1	2	7	8	-5	-6
M_2	4	3	-2	-1	8	-7	6	-5
	5	-6	-7	-8	-1	2	3	4
	6	5	-8	7	-2	-1	-4	3
	7	8	5	-6	-3	4	-1	-2
	8	-7	6	5	-4	-3	2	-1

FIGURE 4. The $\otimes$-multiplication matrix $M_9(K_8)$ = m_{ij}), where $m_{ij} = e_i \otimes e_j = z_{ij} \cdot (e_i \circ e_j)$, which defines the algebra U_3 of order 2^9=8. U_3 is isomorphic to the algebra of Cayley numbers.

associative division algebras are U_2 (quaternions), U_1 (complex numbers) and U_o (F = real numbers); U_3 (commonly known as *Cayley numbers)* is the prototype of the class of *Cayley-Dickson algebras* of order 8. The algebra A_3 defined by the matrix M_3 shown in Figure 3 belongs to this class.

§ 7 – Construction of ⊗-Systems

We shall now show how the ⊗-multiplication matrix $M_r(K_s)$ can be used to construct special finite closed systems such as groups and pseudogroups.

It is clear from Def. 3 that the operation ⊗ defined by $M_r(K_s)$ is not necessarily closed over K_s because of the sign coefficient z_{ij} in its defining equation: $e_i \otimes e_j = z_{ij} \cdot (e_i \circ e_j)$. To form a closed system, it becomes necessary to define ⊗ over a larger set $C(r+1)$ of order 2^{r+1} which contains K_s and elements of the form: $-e_i, \cdot\, i = 1, \ldots, s = 2^r$. Therefore, if we take

$$C = C(r+1) = \{\pm e_i \mid i=1,\ldots,s=2^r\},$$

then the operation ⊗ is closed over C(r+1) such that the following basic relations hold:

$$\begin{aligned} e_i \otimes ej &= (-e_i) \otimes (-e_j) = z_{ij} \cdot (e_i \circ e_j) \\ (-e_i) \otimes ej &= e_i \otimes (-e_j) = z_{ij} \cdot (e_i \circ e_j) \\ -e_i &= (-1) \cdot (e_i, \text{ where } -1 \in F \end{aligned}$$

for all i, j=1,...,s. Any finite closed system of the type <C; ⊗> of order 2^{r+1} shall be called a *⊗-system.* Clearly, $e_i \otimes e_i = e_i^2 = \pm e_1$ for all i=1,...,s. This means that <C; ⊗> contains only elements of orders 1, 2, and 4. Hence any finite closed system which contains only elements of orders 1, 2, and 4 is isomorphic to some ⊗-system of the same order.

The ⊗-system <C; ⊗> of order 2^{r+1} can be explicitly expressed in terms of the matrix $M_r(K_s) = (|m_r|_{ij})$, i, j=1,...s, as follows. Let $S(C) = (\pm|m_r|_{ij})$ be the structure matrix of <C; ⊗>. Partition S (C) into four blocks C_{pq}, p, q = 1, 2, and let $C_{11} = C_{22} = M_r(K_s)$ and $C_{12} = C_{21} = -M_r(K_s)$, where $-M_r(K_s) = (-|m_r|_{ij})$. Then we can write: $S(C) = (C_{pq}) = (\pm|m_r|_{ij})$. This matrix is shown in Figure 4. To indicate that the matrices Z_r and S_r are the building blocks of M_r, and hence of the ⊗-system <C; ⊗>, we shall also symbolically write: $S(C) = Z_r(K_s) \otimes S_r(K_s)$.

$$S(C) = \begin{matrix} M_r & -M_r \\ -M_r & M_r \end{matrix}$$

FIGURE 4. The structure matrix S(C) of the ⊗-system <C; ⊗> shown in block form in terms of the matrix $M_r(K_s)$.

§ 8 – Other Applications

As our first example of a $\otimes$-system, consider the $\otimes$-multiplication matrix $M_r(K_s)$ shown in Figure 3 which defines the Cayley algebra A_r of order 2^r and degree r. Form the set $C(r+1) = \{\pm \ e_i/i=1,\ldots,s = 2^r\}$. Then $<C; \otimes>$ is a $\otimes$-system of order 2^{r+1} which we call the system (*group or pseudogroup*) of *Cayley unit vectors.* If $r \leq 2$, we find that $<C; \otimes>$ is a group (associative); otherwise, if $r \geq 3$, it is a pseudogroup (non-associative).

Next, let $M_u(K_v) = ([m_u]_{ij})$ be a submatrix of $M_r(K_s)$, where $u < r$, $v = 2_u$, and $[m_u]_{ij} = e_r \otimes e_j = z_{ij} \cdot (e_i \circ e_j)$ for all i, $j=1,\ldots,v$. If $C(u+1) = \{\pm e_i \mid i=1,\ldots,v=2^u\}$, then it follows that $<C(u+1); \otimes>$ is a $\otimes$-system of order 2^{u+1} which is a subsystem of $<C(r+1); \otimes>$. Now if u=1, then $u+1 = 1+1 = 2$ and we obtain the $\otimes$-system $<C(2); \otimes>$ of order $2^2 = 4$ which is isomorphic to the cyclic group C_4 of order 4. Moreover, it is also isomorphic to the cyclic group generated by the basis vectors $\underline{1}$ and $\underline{i}$ of the algebra of complex numbers. If u = 2, we obtain the group $<C(3)>; \otimes>$ of order $2^3 = 8$ which is isomorphic to the quaternion group of order 8; this is the group generated by the basis vectors of the algebra of quaternions. And if u = 3, we have the system $<C(4); \otimes>$ of order $2^4 = 16$ which is isomorphic to the pseudogroup of order 16 generated by the basis vectors of the algebra of Cayley numbers which is a non-associative division algebra.

Finally, let us construct two interesting groups $<G; \otimes>$ and $<G^+; \otimes^+>$, both of order $2^{4+1} = 32$, which are isomorphic to the group of *rotations in six dimensions* and the group of *gamma matrices* (or *Dirac operators*), respectively. These groups are involved in such diverse fields as geometry, function theory, and quantum electrodynamics. We shall first construct the group $<G; \otimes>$ and then use it to form $<G^+; \otimes^+>$. To do this, we begin by defining a $\otimes$-multiplication matrix $M_r(G_s) = (g_{xy})$, where r = 4, $s = 2^4 = 16$, and $g_{xy} = g_x \otimes g_y$ for all x, y=1,...,s=16. Since $M_4(\underline{G}_s) = Z_4(\underline{G}_s) * S_4(\underline{G}_s)$, it is necessary to first form the Z-matrix $Z_4(G_s) = (z_{xy})$. Figure 5 shows the required Z-matrix where the indices x, y of the entries z_{xy} are associated with distinct number couples $(a,b) = 4(a-1)+b$, where a,b = 1,2,3,4. Thus if a=b=1, then x = (1,1) = 1 and if a=b=4, then x = (4,4) = 16, etc. With this index notation, the Z-matrices shown in Figure 5 can be defined as follows: $Z_4(\underline{G}_s) = (Z_{xy})$, where

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	–	–	–	–	–	–	–	–
+	+	+	+	–	–	–	–	+	+	+	+	–	–	–	–
+	+	+	+	–	–	–	–	–	–	–	–	+	+	+	+
......				......				......				......			
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	–	–	–	–	–	–	–	–
+	+	+	+	-	–	–	–	+	+	+	+	–	–	–	–
+	+	+	+	–	–	–	–	–	–	–	–	+	+	+	+

....				....				....				....			
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
+	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
+	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
....				....				....				....			
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
+	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
+	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+

FIGURE 5. The Z-matrix $Z_4(\underline{G}_s) = (z_{xy})$, where $x = (i, q) = 4(i-1)+q$, $y = (j,r) = 4(j-1)+r$, and $i,q,j,r = 1,\ldots,4$

$$z_{xy} = z_{<i,q><j,r>} = \begin{cases} +1 \text{ if } q=j \text{ or } q,\ j=1 \\ -1 \text{ if } q\neq j \text{ and } q,\ j\neq 1 \end{cases}$$
$$x = (i, q) = 4(i-1)+q, \quad y = (j,r) = 4(j-1)+r$$

for all x, $y=1,\ldots, S=16$ such that $i,q,j,r = 1,\ldots,4$. Since $S_4(\underline{G}_s) = (e_{xy})$, where $e_{xy} = e_x o e_y$ for all x, $y=1,\ldots,16$, then we finally obtain: $M_4(\underline{G}_s) = Z_4(\underline{G}_s) * S_4(\underline{G}_s) = (g_{xy})$, where

$$g_{xy} = g_x \otimes g_y = z_{xy} \cdot e_{xy} = z_{xy} \cdot (e_x o e_y).$$

This equation completely defines the operation $\otimes$ over the set $\underline{G}_s = \{ei \mid i=1,\ldots, s=16\}$ · See Figure 6.

Now, form the set $G = \{\pm g_i \mid i=1,\ldots,16\}$. Then operation $\otimes$ defined by $M_4(\underline{G}_s)$ is closed over G and the system <G; $\otimes$> is a $\otimes$-system of order $2^{4+1} = 32$. This system is a group isomorphic to the group of *rotations in six dimensions;* it is non-commutative and it contains 1 element of order 1, 19 of order 2, and 12 or order 4. It is not difficult to show that G can be generated by the following set of four elements: $J = \{g_4, g_5, g_{11}, g_7\}$. These elements *anticommute* with each other, that is, $g_a \otimes g_b = -g_b \otimes g_a$ for all $g_a, g_b \in J$. Moreover, $g_4^2 = g_5^2 = g_{11}^2 = g_1$ (order 2) while $g_7^2 = -g_1$ (order 4).

To construct the group <G⁺; $\otimes^+$> of order $2^{4+1} = 32$, we simply form another $\otimes$-multiplication matrix $M_4^+(\underline{G}_s) = Z_4^+(\underline{G}_s) * S_4^+(\underline{G}_s)$ by replacing the Z-matrix $Z_4(\underline{G}_s)$ by the Z-matrix $Z_4^+(\underline{G}_s)$ shown in Figure 7; the substratum remains the same, that is $S_4^+(\underline{G}_s) = S_4(\underline{G}_s)$ which is the Klein group of order $2^4 = 16$. To construct $Z_4^+(G_{-s})$, we modify the set J of independent generators

FIGURE 6. The $\otimes$-multiplication matrix $M_4(\underline{G}_s) = (g_{xy})$. where $g_{xy} = g_x \otimes_{gy} = z_{xy} \cdot (e_x o e_y)$, for all x, $y = 1,\ldots 16$. (Note: $v = g_v$.)

	1	2	3	4	5	6	7	8	9	...	16
M_1	2	1	4	3	6	5	8	7	-10	...	-15
	3	4	1	2	-7	-8	-5	-6	11	...	-14
M_2	4	3	2	1	-8	-7	-6	-5	-12	...	13
	5	6	7	8	1	2	3	4	13	...	12
	6	5	8	7	2	1	4	3	-14	...	-11
	7	8	5	6	-3	-4	-1	-2	15	...	-10
	8	7	6	5	-4	-3	-2	-1	-16	...	9
M_3	9	10	11	12	13	14	15	16	1	...	8
	.	.	.	.	.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.
	16	15	14	13	-12	-11	-10	-9	-8	...	1

of G by replacing its order 4 element, g_7, by another element, g'_7, which is of order 2, to form the new set $J^+ = \{g_4, g_5, g_{11}, g^+_7\}$ and use J^+ to generate the group $\langle G^+; \otimes^+ \rangle$. Again, all elements of J^+ anti-commute with each other but they are now all of the same order 2. The group $\langle G^+; \otimes^+ \rangle$ is isomorphic to the group of *Dirac operators* (or *gamma matrices*); it is non-commutative and contains 1 element of order 1, 11 of order 2, and 20 of order 4. The set J^+ of four anti-commuting elements satisfy the *anti-commutation rules* involved in the *relativistic theory of free electrons* and related matters in *quantum electrodynamics.*

It is important to remark that any system (group, pseudogroup, or algebra) defined in terms of a $\otimes$-multiplication matrix of the type $M_r(K_s) = Z_r(K_s) * S_r(K_s)$ depends on the Z-matrix $Z_r(K_s)$ for its basic characteristics that are not determined by its substratum $S_r(K_s) = \langle K_s; \circ \rangle$. The matrix $M_r(K_s)$ defines the operation $\otimes$ over K_s. Therefore, the system $\langle K_s; \otimes \rangle$ of order s is

+ + + +	+ + + +	+ + + +	+ + + +
+ − + +	+ + + +	− − − −	− − − −
+ + − +	− − − −	+ + + +	− − − −
+ + + +	− − − −	− − − −	+ + + +
....	....	....	....
+ + + +	+ + + +	+ + + +	+ + + +
+ + + +	+ − + +	− − − −	− − − −
+ + + +	− − + −	+ + + +	− − − −
+ + + +	− − − −	− − − −	+ + + +
....	....	....	....

+ + + +	+ + + +	− + + +	+ + + +
+ + + +	+ + + +	− − − −	− − − −
+ + + +	− − − −	+ + + +	− − − −
+ + + +	− − − −	− − − +	+ + + +
....	....	....	....
+ + + +	+ + + +	+ + + +	− + + +
+ + + +	+ + + +	− − − −	− − − −
+ + + +	− − − −	+ + + +	− − − −
+ + + +	− − − −	− − − −	+ + + −

FIGURE 7. The Z-matrix $Z_4^+(\underline{G}_s) + z_{xy}^+)$ used to construct the $\otimes$-multiplication matrix $M_4^+(G_s)$ that defines the group $\langle G^+;\otimes^+\rangle$. This group is isomorphic to the group of *Dirac Operators* (or *gamma matrices*). Note that $Z_4^+(\underline{G}_s)$ differs only from $Z_4(\underline{G}_s)$ in some of its diagonal entries.

not, in general, closed except in the special case when $Z_r(K_s) = Z_1$, where Z_1 is the sxs *unit Z-matrix*. The order of an element of $\langle K_s;\otimes\rangle$ is determined by the equation: $e_i \otimes e_i = e_i^2 = z_{ii} \cdot e_1$. If $z_{ii} = +1$, e_i is of order of 2 or 1 (only e_1 is of order 1), and if $z_{ii} = -1$, e_i is of order 4. This means that the diagonal entries z_{ii}, of $Z_r(K_s)$ determine the number of elements of orders 1, 2, and 4 in $\{K_s; \otimes\}$ Now, if $z_{ij} = z_{ji}$ for all i, j=1,...,s, then it follows that $e_i \otimes e_j = e_j \otimes e_i$ for all $e_i,e_j \in K_s$ and hence $\langle K_s;\otimes\rangle$ is commutative. Otherwise, if there is at least one pair of entries z_{ij} and z_{ji} such that $z_{ij} \neq z_{ji}$, then it follows that $\langle K_s; \otimes\rangle$ is non-commutative.

The use of the $\otimes$-multiplication matrix in the construction of algebras and $\otimes$-systems is indeed a fruitful one. This is because there is a large number of possible Z-matrices to choose from in forming a desired matrix $M_r(K_s)$ of dimensions sxs. The required Z-matrix is in the set Z(sxs) which is of order 2^{sxs} and, if, for instance, $s = 2^2 = 4$, then we have a total of $2^{4x4} = 65{,}536$ possible 4x4 Z-matrices!

As a last remark, we state that the *$\otimes$-multiplication matrix* can be generalized to mean any nxn matrix $M(E) = (e_{ij})$, where $e_{ij} = e_i \otimes e_j$, i,j=1,...,n, $\otimes$ is any binary operation over E, and $E = \{e_i | i=1,\ldots,n\}$ is any set of n elements. Here, the binary operation $\otimes$ need not be closed over E. In this generalized sense, the matrix $M_r(K_s)$ given by Def. 3 is seen to be just a special form of $\otimes$-multiplication matrix. If $\otimes$ is closed, however, M(E) reduces to a simple *structure matrix*, that is, M(E) = S(E).

§ 9 – Summary

The *Z-matrix* is a special kind of matrix whose entries are the symbols + and – representing the numbers +1 and −1. This matrix has interesting and useful properties with important applications in abstract algebra and in modern theoretical physics. Z-matrices exist in all dimensions mxn and they form commutative groups of order 2^r under the operation * of *star multiplication.* Every Z-matrix group <Z;*> of order 2^r is isomorphic to a Klein group of the same order; hence every Klein group of order 2^r can be *represented* by a Z-matrix group of the same order and dimension mxn such that mxn $\geq$ r.

Z-matrices can also be used to form *division algebras* as well as special *pseudogroups* and *groups.* This is made possible by means of the ⊗*-multiplication matrix* $M_r(K_s) = Z_r(K_s)*S_r(K_s)$, where $Z_r(K_s)$ is an sxs Z-matrix and $S_r(K_s)$ is the sxs structure matrix of the Klein group $<K_s;o>$ of order $s = 2^r$. By a suitable choice of $Z_r(K_s)$ we can construct a family of division algebras A_r of order 2^r (called *Cayley algebras of degree r*) which includes the subalgebras A_1, A_2, and A_3 isomorphic to the *complex numbers,* the *quaternions,* and *Cayley numbers,* respectively. $M_r(K_s)$ can also be used to construct the ⊗-system family of finite closed systems of order 2^{r+1}. This family includes such important systems in theoretical physics as the group of *rotations in six dimensions,* the group of *Dirac operators,* and the pseudogroup of *Cayley basis vectors.* All systems (algebras and ⊗-systems) defined by means of ⊗-multiplication matrices of the type $M_r(K_s)$ have a common substratum: the *Klein group of degree r.*

Postscript

Finally, Dr. Rene Felix of the University of the Philippines in Diliman has pointed out that the Z-matrix is similar to the *Hadamard Matrix H* used in coding theory and statistics. A short review of the literature has shown that indeed H is a special form of nxn Z-matrix, where n = 1, 2 or a multiple of 4, with mutually orthogonal rows. Thus, if H^t is the transpose of H, then $HH^t = nI$, where I is the nxn unit matrix. Moreover, the *star product* of two matrices defined in § 3 of this paper is also known as the *Hadamard product.*

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