THE MATHEMATICS OF SINGULARITIES[1]

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ABSTRACT

Set-valued function as a singular function was used in 2, 3, 4, 5, 7, 8 and 9 to introduce the generalized integral with applications to differential equations and quantum mechanics. A calculus of set-valued functions was developed in 6 and appplied to the wave packet, path integration and superconductivity of quantum mechanics.

This paper considers mathematical singularities and broadens the notion to include contradictions, paradoxes and mathematical dead ends. Then it focuses on the Lebesgue paradox and indicates mathematical applications to physical singularities towards a new theory of gravitation.

(The author got hold of a book on superstring theory while the present paper was still being encoded. It was a pleasant surprise to know that the mathematics he had been doing in the last two years, including this paper, fits snugly into the theory. However, rather than upgrade this paper with the new information, he retained it as originally written to serve as a chronicle of the development of new ideas. A sequel, "Some Mathematical and Physical Principles of Superstring Theory" was quickly written and completed even before this one.)

1. Singularities

In classical mathematics, discontinuities are usually teasers and sources of paradoxes. They are generally avoided but when that is not possible, elaborate schemes are used to "tame" or at least make them manageable. In this paper essential discontinuities are reclassified into three kinds: a jump; infinity, plus or minus; and set-valued.

Discontinuities are special cases of singularities that occur in more general spaces of functions, especially in differential equations and complex analysis. Some singularities have been studied mainly to overcome their "nastiness". Thus jump discontinuities appear in differential equations, especially in connection with the Laplace transform or the heat equation. The second type also appears in the calculus of poles and residues in complex analysis. Some nice theorems about them exist, such as the Casorati-Weierstrass and Picard theorems about isolated singularity but they refer to the neighborhood of a singularity. Singularities of the second type open up possibilities for extending the Laplace transform to nonconvergent improper integrals with the introduction of measure distributions. That would extend the generalized integral to unbounded oscillation such as integrals of the type

$$\int_0^{\pi/2} \tan x \, dx \quad \text{or} \quad \int_{-\varepsilon}^{\varepsilon} \tan\frac{1}{x} dx$$

(See [6] for discussion of the generalized integral).

A study can be made on integrals of the type $\int_0^{\pi/2} \tan x \, dx$ by introducing a suitable measure distribution $\dot{p}(\cdot)$ that would shrink the integrand and results in a convergent improper integral. Of course, there are always trivial distributions that can make this particular integral converge, such as $\pi_{/2} - x$ or $\cos x$. Thus,

$$\int_0^{\pi/2} \tan x \cos x \, dx = 1 \quad \text{and} \quad \int_0^{\pi_{/2}} (\pi_{/2} - x)\tan x \, dx = 1.$$

A more interesting distribution for this integral is

$$\exp\left[\frac{-a}{\left[\frac{n}{2} - x\right]^n}\right], a > 0,$$

where n is an even positive integer.

As $x \to \pi_{/2}$, this function tends to zero while tan x tends to ∞ so that the question of convergence of the integral,

$$\int_0^{\pi/2} \exp\left[-\frac{a}{\left[\frac{\pi}{2} - x\right]^n}\right] \tan x \, dx$$

is not trivial. This idea has applications in operations research or in setting up a factory complex where the price of real estate depends on proximity to the highway or trading exchange facility.

The integral

$$\int_{-\varepsilon}^{\varepsilon} \tan \frac{1}{x} \, dx$$

can be the springboard for studying unbounded generalized oscillation 7 with finite measure since the integrand is set-valued at $x = 0$, its values being the set of real numbers. Then the appropriate measure distribution would be the generalized derivative of its set- value with respect to this measure(6). As in 3, a differential equation of the type

$$\dot{y} = \tan \frac{1}{x}$$

can also be approximated by a wild generalized oscillation with this measure distribution which can be normalized to a probability distribution.

The third type of discontinuity is not as well studied but those working on classsical integration try to shrink it to a point to achieve convergence. Of course, the resulting shrunk function is something else. For instance, the topologist sine curve $f(x) = \sin^n \frac{1}{x}$ is set-valued at $x = 0$. Some mathematicians have studied the integrability of $x^2 \sin^n \frac{1}{x}$ which, certainly, is a convergent improper Riemann integral since it has removable discontinuity at $x = 0$.

The perspective in this paper, however, is entirely different. Singularities are neither avoided nor tampered with to make them benign. The full force of their "nastiness" is welcomed, and some mathematical theories are built out of them and physics. With this perspective, singularities are a gold mine of ideas out of which some useful mathematical theories can be built.

However, the meaning of singularities is first broadened to extend to places at which classical mathematics breaks down. Included in this category are the essential singularities of complex analysis, set-valued functions, and Peano's space-filling curves. Also included are the contradictions, paradoxes and dead ends of classical mathematics.

The dead ends are those areas of classical mathematics that cannot advance further wthout radical alteration of the basic concepts there or a fundamental reformulation of the problems being addressed by those areas. For example, the Lebesgue integral is just about at the zenith of its evolutionary journey. It can deal with well-behaved functions alone, such as those with simple discontinuities including jumps or measurable and essentially bounded functions, such as some common improper integrals. However, it breaks down when confronted with a

function that tends towards a set such as $\sin^n \frac{1}{x}$. Thus, this particular pursuit reaches a dead end at that point in the long historical development of the integral. To break up that dead end and open up a new path beyond it, new concepts had to be introduced and a new theory build that would encompass all the stages in the development of the integral. In this case the essential concepts needed are the generalized notions of limit and continuity, the generalized integral and derivative and generalized oscillation and pulsation. These notions were developed in several references(3, 4, 5, 6, 7, 8 and 9) which paved the way for a new theory – the calculus of set-valued functions.

Another dead end that has dissipated a lot of mathematical energy is Fermat's conjecture: that there are no integers x, y and z satisfying the equation $x^n + y^n = z^n$, where n is an integer greater than 2. Here, a new orientation is needed to achieve some breakthrough: to study two types of axiomatic systems, one that includes this conjecture as an axiom and another that excludes it. That would be analogous to the development of non-Euclidean geometry.

Some dead ends are inadequacies. For example, quantum mechanics needs special mathematics for probabilistic motion. It would involve set-valued functions (8). Present path integration there is also flawed; it does not make sense for the motion of an elementary particle. It is proposed that the locus of the expectation point be taken in place of that "path" but this would require a calculus of set-valued functions with appropriate probability distribution function of the set-values of such function. That is the kind of program that would lead to further breakthroughs in the understanding of the wave packet.

An important fallacy whose resolution led to an important breakthrough in Analysis is the use of necessary condition without an existence theory. This was resolved by Young (14) in 1937 for the calculus of variations by building up an existence theory. In the proccess, the theory of generalized curves was developed. A generalized curve is the solution of a differential equation with set-valued derivative; and it is shown in 14 that it is the fine limit of a sequence of traditional curves. Unfortunately, this fallacy still remains unnoticed in many parts of mathematics today. For example, in dealing with the heat equation of partial differential equations, one starts with the statement, "Let the solution be $T(x, y) = g(x) h(y)\ldots$" (separated variables), without raising the question of whether such a nice solution exists in the first place. Godel's incompleteness theorem appears to be a paradox or, at least, a bit unsettling; it is a difficult question that may not be resolvable in mathematics proper. It is a philosophical question that might have something to do with the appropriate logic for our present level of knowledge. Godel's incompleteness theorem was proved with deterministic logic.

We can go on and on but for the optimist – for one who sees a gold mine in a desert of contradictions — this only augurs well for mathematical research.

2. Singular Functions

As for first example of singularities of the third type, we refer to the calculus of set-valued functions. A formal theory is developed in 6. It is the appropriate mathematics for probabilistic motion which is immediately applied to some fundamental problems of physics, especially in the study of the wave packet and path integration in quantum mechanics. It contains the basic ingredients of a calculus: generalized notions of limit and continuity and the generalized integral, derivative and oscillation. A more general treatment of generalized integration differentiation and pulsation is given in 3, 4 and 7. This theory is open to further research; in particular, the counterparts of the important theorems of elementary calculus, such as the mean value theorems for the integral and derivative, have yet to be developed.

An integro-differential geometry based on the generalized integral and derivative can be developed as well. That would be a modification and extension of the formulation in 3. Instead of introducing a probability distribution on the control set as done in 15, we introduce a probability distribution on the set values of the derivative at each point (t,x). For example, suppose the set-values of the derivative function at the point (t,x) in R^{n+1} are given by

$$(1) \qquad\qquad g(t, x, \Omega_{tx}),$$

where Ω_{tx} is a compact set in the vector field of direction vectors and the set-values of g lie in R^{n+1}. Suppose, further, that we introduce a probability distribution function $\dot{p}_{tx}$ on the set value $g(t, x, \Omega_{tx})$, then the behavior of the trajectory at the point (t, x) can be described by the quadruple $(t, x, g(t, x, \Omega_{tx}), \dot{p}_{tx}(\cdot))$ and we can set

$$(2) \qquad\qquad \dot{x} = \int_{\{g_{tx}\}} (\cdot)\, dp_{tx}(\cdot) = E(t, x), \text{ a.e.,}$$

where $\{g_{tx}\}$ is the set-value of g at the point (t, x) and $E(t, x)$ is the expectation point. Thus, equation (2) can represent the equation of motion of a probabilistic dynamical system. The idea is to transform first the set-valued function g into the well-defined expectation function $E(t, x)$.

Of course, we must assume some measurability conditions to insure existence. Uniqueness is not necessary for purposes of qualitative mathematics; in fact, more information can be gained from the space of solutions and the study of its structure. However, in situations where uniqueness is needed, some Lipschitz condition on x for each value of g would suffice. In that case, given the initial condition $x(t_0) = x_0$, we can express the solution of Equation (2), as

$$(3) \qquad\qquad x(t) = x_0 + \int_{t_0}^{t} \left(\int_{\{g_{sx}\}} (\cdot)\, dps(\cdot)\right) ds,$$

using a standard technique in differential equations.

Now we focus on another singularity which used to be a paradox. It was first raised by Lebesque and used by Young to spur the development of generalized curves – a major achievement in mathematics that resolved the Perron paradox 15 and opened up many new fields of mathematics.[1] We use it here for a different purpose.

Let s be any line segment and denote its length by |s|. Let AB be a line segments of length |AB| < |s|. Then s can be deformed continuously to form two sides AD and DB of triangle ABD (Figure 1). Join the midpoint P of AD to the midpoint Q of AB and the point Q to the midpoint R of DB as shown. The segments PQ and QR are parallel, respectively, to the sides DB and AD of the triangle. From the geometry of the figure, the length of the polygonal line APQRB is equal to the length |s| of line segment s which now forms the two sides of AD and DB of triangle ABD. Repeat this construction to obtain, at the nth step, a polygonal line C_n from point A to point B whose length is also |s|. Let n — ∞. Then the polygonal line C_n approaches its set limit or projection limit AB. Let that limit be Γ. It is clear that Γ coincides exactly (pointwise) with side AB but its length is given by

$$\lim_{n \to \infty} |C_n| = \lim_{n \to \infty} |s| = |s|$$

which is distinct from |AB|; in fact, |s| > |AB|.

Figure 1

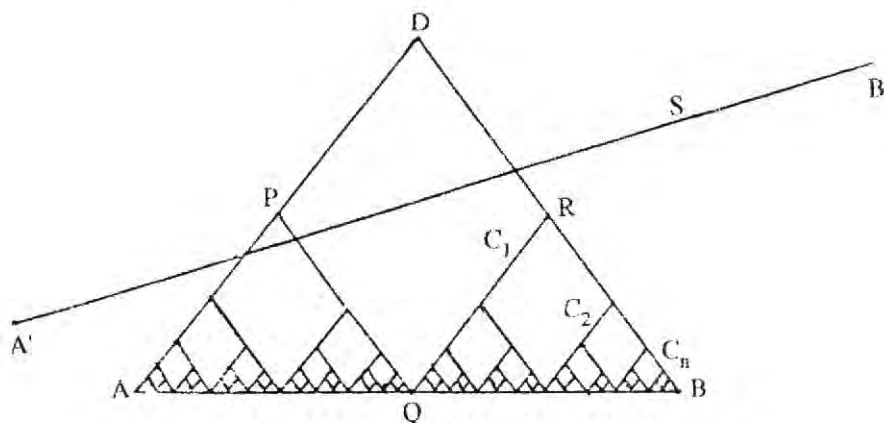Since the real number |s| is an arbitrary number greater than |AB|, we have another case of a set-valued function here but this time a set-valued function on sets (see reference 6 for functions on sets). In fact, given any real number r > |AB|, there exists a polygonal line of length r that tends to AB pointwise and whose length remains r. We can look at the segment AB as an infinity of distinct but coincident curves, distinct because their lengths are distinct. It is clear that the cardinality of the sets of lengths of these coincident curves is ψ1, the cardinality of the continuum. We define a generalized notion of length L(AB) of AB as set [|AB|, ∞), a half open extended interval, where |AB| is the ordinary length of the segment AB. Thus the infimum of the set-value of L(AB) is the ordinary length |AB| of AB. Any number r E [|AB|, ∞) is a particular value of the generalized length L(AB). It is clear that only a curve or line segment in the plane with ordinary length |s| ≥ |AB| can be deformed in this manner, with length preserved, to coincide with the segment AB whose ordinary length is |AB|.

In this construction the choice of AB is arbitrary. Hence, given ε > o, we can deform any rectifiable curve C of any ordinary length r ≥ ε, no matter how large, into a neighborhood S of some point of diameter d (S) < ε, with its original ordinary length preserved. Let |AB| = β < ε and r ≥ ε. We first deform C into two sides AD and DB of isosceles triangle ABD, where |AD| + |DB| = r and |AB| = β and proceed with the same construction above. We proceed far enough, that is, we let n be large enough that the polygonal curve C_n would lie in the ε-neighborhood of AB in the supremum norm. By construction, the length of C is preserved and is equal to the length $|C_n|$ = |C| at the nth step in the construction.

The moral of the story here is that any curve of any length can be deformed, with its length preserved, to fit into an arbitrarily small neighborhood of a segment or a point.

Since ordinary length can be a vector representing a measure of any kind – mass, energy, velocity, etc. – this construction has a lot of implications: it is possible to compress a great amount of matter or tremendous energy or a great mass into an infinitesimal element of space that is not detectable by electromagnetic means due to differences in orders of magnitudes both in terms of energy level as well as frequency and wave length in the case of an oscillatory dynamical system. (A sequel, "Some Mathematical and Physical Principles of Superstring Theory" has an elaboration on this important point).

Examples of such great concentration of energy are the energy trapped in the nucleus of an atom and the black hole. Neither of them is directly detectable by electromagnetic means. A recent conjecture by Hawking 1 says that electromagnetic waves escape from a black hole where gravitational force is dominant. We assert here that what is referred to as strong nuclear forces or binding force in the nucleus of an atom is the result of interaction among the building blocks

of matter, the same interaction responsible for gravitation, at a higher energy range above that of the electromagnetic range by several orders of magnitude. We will deal with this matter elsewhere.

For our first application of this idea, we consider an oscillation of the type $f(x) = \sin^n w$. Oscillation is a universal phenomenon in physics and is the key to an understanding of important physical singularities. The energy of an oscillation is given by $h\upsilon$ where h is the universal Plank's constant and υ is the frequency. Assume for the moment that matter consists of high-frequency oscillatory strings which we shall represent by oscillatory curves. Then the total energy of an oscillatory system (oscillatory curve) is proportional to its length.

Given an oscillatory curve K, which is rectifiable, it can be deformed, with its length preserved, into the two sides of an isosceles triangle ABD on base AB. There exists an oscillatory curve from A to B consisting of two wave lengths whose length is equal to that of the polygonal line APQRB of the above construction shown in Figure 2. Replicate this construction on the triangle APQ and QRB to be able to construct an oscillatory curve K_2 from A to B consisting of four wave lengths, each of which corresponds and is equal in length to the combined length of two sides of an isosceles triangle of the polygonal line at that stage in the construction shown in Figure 2. Thus, we have a finer oscillatory curve from A to B whose length is equal to that of C. Continuing, we obtain, at the nth step, an oscillatory curve K_n from A to B whose length is equal to that of C_n in the construction shown in Figure 2 and, therefore, also equal to that of K. K_n tends to AB pointwise as $n \to \infty$. Since the length of AB is arbitrary, we can choose it to be of length less than ε, $\varepsilon > 0$, and so we can shrink an oscillatory curve, with length preserved, into an arbitrarily small neighborhood of a segment or a point. The construction is shown in Figure 2.
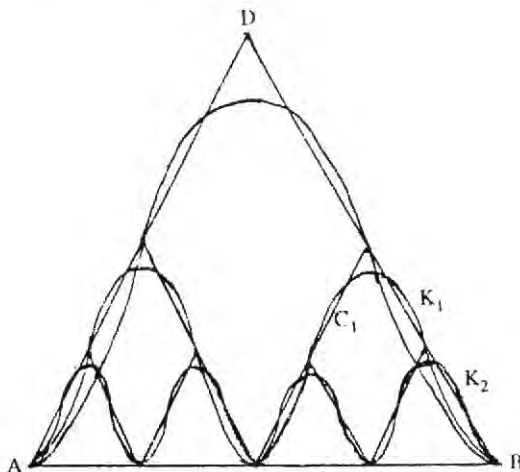

Figure 2.

The limit of such a curve along a segment is a generalized curve. Such an infinitesimal oscillation, for sufficiently large n, would not interact with macro-oscillation such as an electromagnetic wave because of the difference in orders of magnitude in terms of both frequency and wave length. With this construction, it is possible to store a great amount of energy in an infinitesimal oscillation without being detected by electromagnetic means. We state this as a Theorem.

Theorem. Given an oscillatory curve K, any number $\varepsilon > 0$ and a line segment AB, there exists a continuous deformation of K into a fine oscillatory curve K inside some ε-neighborhood of AB which preserves the length |K| of the curve K.

Since the length |AB| can be chosen arbitrarily small, we also have the following theorem:

Theorem. Given any oscillatory curve K, there exists a continuous deformation of K, with length preserved, into an arbitrarily small neighborhood of a point.

Proof. Let A be a given point in the plane, B a point in the ε-neighborhood of A and suppose $|AB| = \beta > 0$. There exists a deformation of K into two sides AB and DB of an equilateral triangle ABD where $|AD| + |DB| = |K|$. Following the construction above there exists a sequence of polygonal curves C_n and a corresponding oscillatory curve K_n such that for each n, $|K_n| = |C_n| = |AD| + |DB|$ and K_n tends to the segment AB. Hence there exists a positive integer N such that whenever $n \geq N$, the curve K_n would lie inside the ε-neighborhood of A.

Note that in each case the oscillatory structure is preserved as well as its length. Thus it is possible to shrink an oscillatory curve of any length into an infinitesimal oscillation at a point.

Now, let $\beta > 0$, where β is small and let K be an oscillatory curve of large length |K|. Let $\varepsilon = \beta/2 < \frac{|K|}{2}$. As before, we can deform K into the two sides of an isosceles triangle ABD with base AB, where $|AB| = \varepsilon$. Let h be the altitude of this triangle. Then h is roughly |K|/2. By the Archimedian property of the reals, there exists some positive integer n such that

$$(4) \qquad \frac{|K|}{2^{n+2}} < \frac{|K|}{2^{n+1}} \leq \varepsilon < \frac{|K|}{2^n}$$

Therefore, in the sequence of oscillatory curves K_i with $|K_i| = |K|$, for each $i = 1, 2, \ldots$, which tends towards the line segment AB, there is one whose amplitude satisfies the inequality (4). We state this as a theorem.

Theorem. Let K be an oscillatory curve with large length |K| and let $\beta > 0$, $\varepsilon = \beta/2 < \frac{|K|}{2}$. Then one can shrink the oscillatory curve K into an arbitrarily small neighborhood of a point with its length preserved and its amplitude prescribed to satisfy

$$\frac{|K|}{2^{n+2}} < \frac{|K|}{2^{n+1}} \leq \varepsilon < \frac{|K|}{2^n}$$

for some integer n.

So far, the oscillatory curves we have considered are ordinary oscillations, i.e, periodic, and their equation is given by $y = \sin^n x$, where n is a positive integer. The construction holds for any rectifiable curve (e.g., absolutely continuous curve). In particular, we can allow the frequency υ to progressively increase in accordance with the above deformation. The curve that will do this is the topologist sine curve given by

$$y = \sin^n \frac{1}{x} ,$$

where n is a positive integer. This has the property that it can pack an oscillatory curve of any length into an ε-neighborhood of $x = 0$. This has been considered in (6).

We shall consider elsewhere the geometry of oscillations. Suffice it to say at this point that if we think of a Universe where small units of matter or energy are in oscillatory motion, especially infinitesimal motion, then these units can be distinguished by the geometry of their oscillations in terms of amplitude, frequency and structure some of which may exist infinitesimally in higher dimensions. Some such infinitesimal dynamical systems may be stationary or slow-moving, others fast moving, and still others may have almost infinite speed. We also allow all possible distortions of the oscillatory motion even as far as set-valued limits in which case we enter the subject considered in (6). If they are at high energy levels but in infinitesimal motions, they are undetectable in their normal states by other forces, such as electromagnetic energy, due to difference in orders of magnitude. Here lies the key to an understanding of gravitation which will be discussed elsewhere.

CONCLUSION

We conclude this article with some observations:

1.   Both Relativity and Quantum mechanics are descriptive theories and reduce motion to geometry in terms of curvatures or structures described by equations and functions. While such a description is highly accurate at this time and has predictive value, neither Relativity nor Quantum mechanics explains why nature behaves this way. In that sense both are a bit mystical and neither of them is a dynamic theory.

2. In the world of Relativity and Quantum Mechanics, existence is discrete – that is the implication of the quantization principle. That the Universe continues to exist with a certain level of continuity and regularity implies that there is a hidden or invisible region – hidden from both theories – that provides

the continuity. In fact, physicists have raised the issue of hidden or dark matter composed of at least 90% of the mass of the Universe that is unaccounted for at this time.

3. The traditional notion of gravity by Relativity does not explain action at a distance across empty space; Relativity only describes it.

4. The tremendous energy stored in the nucleus of an atom can be explained by clusters of such highly energetic infinitesimal oscillatory material which we can take as the building blocks of matter.

5. Resonance caused by the entry of a neutron at the right energy state or abrupt change in the electromagnetic field could unscrew such stationary infinitesimal oscillation leading to the splitting of the nucleus and release of tremendous energy; such energy can be released also by abrupt change in potential or voltage such as what triggers a volt of lightning.

6. The discrete existence of a subatomic particle in flight, in accordance with the quantization principle of quantum mechanics, could be due to fluctuations in energy level of the clusters of energy units in flight; the discrete existence of a vector boson coincides with the wave packet of the corresponding energy unit in flight and its observed discrete existence is also due to fluctuations in orders of magnitude of their oscillations.

7. The fluctuations in energy levels of the vector boson towards the undetectable level could explain the loss of electromagnetic interaction and other forms of resistance and energy dissipation which could in turn, explain further the phenomenon of superconductivity. Here we raise the possibility of transfer of energy without a carrier in the visible region of matter. This implies the existence of a carrier at the invisible region which therefore would not interact with electromagnetic entities.

8. Radiation from an atom is oscillation that has been reduced to the electromagnetic range.

9. Gravitation is energy at a higher range with the energy field as its medium; it does not interact with other forms of energy at different orders of magnitude both in terms of frequency and amplitude; this ties up with Hawking's conjecture that at an event horizon of a black hole, where gravitation has taken dominance there is radiation escape. If matter consists of fundamental building blocks, then gravitation is interaction at this level of the constituent of matter and that is how it acts on larger configuration such as huge masses.

10. Observation of the spiral Galaxy suggests viscosity on the part of the energy field. It is also elastic as it allows propagation of fine oscillations. This viscosity also explains the shift in the orbit of Mercury, an anomaly in the Solar system that has puzzled physicists, including relativists. It is the net effect of both the absorption of the energy field by the sun and its rotation.

11. A new gravitational theory which has not been formally and mathematically formulated yet has emerged, based on an assumption of an energy-rich field between masses rather than an essentially empty space between them.

This formulation will be dealt with in another study.

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