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CONSTRUCTION OF ALL CAYLEY ALGEBRAS OF ORDER 2^r BY THE ZSM PROCESS

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ABSTRACT

The existence of Cayley Algebras of order 2^r is established by construction. These are real division algebras which include the real numbers R (order 2^0), the complex numbers C (order 2^1) and the quaternions H (order 2^2) all of which are associative – and the Cayley numbers O (Order 2^3) which are nonassociative. This paper shows that all of these real division algebras have a common structure exemplified by the Cayley numbers and they all belong to a single family composed of classes of Cayley algebras of order 2^r, where r is any positive integer. This is done by introducing the ZSM Process to construct all of these algebras.

INTRODUCTION

In 1845 A. Cayley constructed a remarkable *real division algebra* of order 8 (now known as the *Cayley numbers* O) which is *nonassociative, noncommutative, normed* and contains as subalgebras the *quaternions* H (order 4), the *complex numbers* C (order 2) and the *real numbers* R (order 1) itself. G. Frobenius proved in 1878 that the only real associative division algebras of finite order are H, C and R, all of which are normed. Attempts to determine other normed real algebras of finite order led A. Hurwitz in 1898 to the theorem that the only algebras of this type are of orders 1, 2, 4 or 8. In 1947, A. Albert showed that these are again R, C, H and O. Then in 1957, R. Bott and J. W. Milnor finally proved that the only finite dimensional real division algebras are of orders 1, 2, 4 and 8. Pursuing more general considerations, L. Dickson introduced in 1923 a general method (called the *Cayley-Dickson Process*) and used it to construct the class of order 8 real division algebras which includes O as its prototype. This paper shows that all real division algebras of order 2^r (like R, C, H and O) belong to a family of classes of *Cayley algebras of order* 2^r. This interesting family is constructed by introducing the ZSM Process thereby proving the theorem that: *There exists a class of Cayley algebras of every oder 2^r, where r is any positive integer.*

DIVISION ALGEBRAS OF ORDER 2^r

Consider the algebra $A_r = \{V, F; +, x, \otimes, \otimes,.\}$ over the field $F = R$ of real numbers. Take as the basis of the n-dimensional vector space V the set $E_n = \{e_i/i=1,\ldots,n\}$ of n *basis vectors* over which the binary operation $\otimes$ is defined by the $\otimes$-*matrix* $M_r(E_n) = (m_{ij})$, $i,j = i,\ldots,n$, where

$$m_{ij} = e_i \otimes e_j = z_{ij} \cdot e_k, \qquad \text{Eq. (1)}$$

$e_i, e_j, e_k \in E_n$, and $z_{ij} \in F$. Every vector of this algebra can be expressed uniquely as a linear combination of the n basis vectors in E_n. Thus, if a. $b \in A_r$, then

$$a = \sum_{i=1}^{n} a_i \cdot e_i \text{ and } b = \sum_{j=1}^{n} b_j \cdot e_j \qquad \text{Eq. (2)}$$

where $a_i, b_j \in F$. *Vector multiplication* is defined by *bilinearity* and the matrix $M_r(E_n)$ so that the product $a \otimes b$ of any two vectors $a, b \in A_r$ is given by the expression

$$a \otimes b = \sum_{ij=k}^{n} f_{ij} \cdot (e_i \otimes e_j) = \sum_{ij=k}^{n} f_{ij} z_{ij} \cdot e_k \quad (k=1,\ldots,n) \qquad \text{Eq. (3)}$$

where $f_{ij} = a_i b_j$ and the index $ij=k$ means that the sum is to be extended over all pairs of indices i,j for which the relation holds: $e_i \otimes e_j = z_{ij} \cdot e_k$. This can be expanded into

$$a \otimes b = \sum_{ij=1} f_{ij} z_{ij} \cdot e_1 + \sum_{ij=2} f_{ij} z_{ij} \cdot e_2 + \ldots + \sum_{ij=n} f_{ij} z_{ij} \cdot e_n \qquad \text{Eq. (3.1)}$$

By definition, an algebra A_r over a field F is a *division algebra* if it has a *unity* for vector multiplication and every non-zero vector $a \in A_r$ has a unique *inverse* $a^{-1} \in A_r$, that is, $a \otimes a^{-1} = a^{-1} \otimes a = e_1$, where e_1 is the unity of $\otimes$-multiplication. Such a vector a^{-1} exists in A_r if a vector a^*, called the *conjugate* of a, can be defined such that $a \otimes a^* = a^* \otimes a = (N(a) \cdot e_1$, where N(a), called the *norm* of a, is a positive element of the field F. If such a vector a^* can be defined in A_r, then $a^{-1} = a^*/N(a)$ fulfills all the requirements of an inverse of $a \in A_r$.

To determine the necessary and sufficient conditions for the inverse a^{-1} of a to exist in A_r, first form the products $a^* \otimes a$ and $a \otimes a^*$ by means of Eq.(3.1). For a^* to be the conjugate of a,

$$a \otimes a^* = a^* \otimes a = \sum_{i\,j=k} f_{ij} z_{ij} \cdot e_k = \begin{cases} N(a) \cdot e_1 & \text{if } e_i \otimes e_j = z_{ij} \cdot e_1 \\ \text{zero} & \text{if } e_i \otimes e_j = z_{ij} \cdot e_1 \end{cases} \qquad \text{Eq. (4)}$$

where $N(a) = \sum_{i\,j=1} f_{ij} z_{ij}$ (summed over all i,j for which $e_i \otimes e_j = z_{ij} \cdot e_1$), $f_{ij} = a_i a_j^*$, and a_i, a_j^*, are the field coefficients of a, a*. These equations constitute the necessary and sufficient conditions for a^{-1} to exist in A_r. Any vector a* that satisfies Eq. (4) is a conjugate of a. If $a^* \in A_r$, then it follows that $a^{-1} \in A_r$.

Consider once more the three well known real division algebras: the *Cayley numbers* O (order 2^3), the *quaternions* H (order 2^2) and the *complex numbers* C (order 2^1). Since O contains H and C as subalgebras, then they all share a number of basic properties in common:

1. They all have orders (or dimension) of the form $n = 2^r$, where r= 1,2,3.
2. Their basis vectors $e_i \in E_n$ satisfy the following set of fundamental equations:

$$\begin{array}{lll} e_i \otimes e_i = e_i^2 = -e_1 & (\text{if } i \geq 2) & \\ e_i \otimes e_1 = e_1 \otimes e_i = e_i & (\text{for all } i) & \text{........ Eq. (4)} \\ e_i \otimes e_j = -e_j \otimes e_i & (\text{if } i \neq j,\ i,j \geq 2) & \end{array}$$

3. Any vector $a = a_1e_1 + a_2e_2 + , \dots, + a_ne_n$ $(a \neq O)$ has a conjugate a*, a norm N(a), and an inverse a^{-1} given by

$$a^* = a_1e_1 - (a_2e_2 + \dots + a_ne_n) \qquad \text{Eq. (5)}$$

$$N(a) = a_1^2 + \dots + a_n^2 \qquad \text{Eq. (6)}$$

$$a\text{-}1 = \frac{a^*}{N(a)} \qquad \text{Eq. (7)}$$

These properties are clearly exhibited by the matrix $\mathfrak{M}_3$ shown in Figure 1 which defines an algebra $\mathfrak{U}_3$ isomorphic to O. Here, the submatrices $\mathfrak{M}_1$ and $\mathfrak{M}_2$ define algebras isomorphic to C and H, respectively. Moreover, if the sign coefficients of the entries of $\mathfrak{M}_3$ are separated into another matrix Z_3 (E_8), then the resulting matrix $S_3(E_8)$ can be seen to have the structure of the Klein group $\langle E_8; o\rangle$ of order $n = 2^3$ shown in Figure 2. Note that $\mathfrak{M}_1$, $\mathfrak{M}_2$ and $\mathfrak{M}_3$ have the form $\mathfrak{M}_r = (m_{ij})$, where $m_{ij} = e_i \otimes e_j = z_{ij} \cdot e_k$, e_i, e_j, e_k represents basis vectors, and $z_{ij} = \pm 1$ are sign coefficients. This means that $\mathfrak{M}_r$ is simply the matrix representation of Eq. (4), where r = 1, 2, 3. These equations, however, do not completely define the operation $\otimes$ over the basis vectors in E_n. Rather, they constitute a set of *necessary conditions* that define a class of algebras of which the Cayley numbers are the protoype

The conditions given by Eq. (4) for any $n = 2^r$, where r is any positive integer can be generated to form a ⊗-matrix $M_r(E_n)$ such that Eqs. (5), (6) and (7) hold. To do this, introduce two special matrices $Z_r(E_n)$ and $S_r(E_n)$ of the same dimensions nxn which shall be called the *sign matrix* and *structure matrix*, respectively. The sign matrix is defined as: $Z_r(E_n) = (z_{ij})$, $i, j = 1, \ldots, 2^r$, where $z_{ij} = \pm 1 \in F$ (the real numbers +1 and −1). On the other hand, the structure matrix is defined as: $S_r(E_n) = e_{ij})$, $i,j=1,\ldots,2^r$, where $e_{ij} = e_i o e_j$, which defines the *abelian p-group* $<E_n;o>$ of order 2^r (where $e_i^2 = e_1$ for all $e_i \in E_n$ and e_1 is the identity element) which shall be called the *Klein group* of order 2^r. Next, introduce the *star product* * of any two n X n matrices $A = (a_{ij})$ and $B = (b_{ij})$, as the n X n matrix $A*B = (c_{ij})$, $i,j=1,\ldots,n$, where $c_{ij} = a_{ij}b_{ij}$. Now, form the star product of $Z_r(E_n)$ and $S_r(E_n)$ obtaining: $Z_r(E_n)*S_r(E_n) = (c_{ij})$, $i,j=1,\ldots,2^r$, where $c_{ij} = z_{ij} \cdot e_{ij} = z_{ij} \cdot (e_i o e_j)$. If we let $Z_r(E_n)*S_r(E_n) = M_r(E_n)$ and set $m_{ij} = c_{ij}$. Then write:

$$\left.\begin{array}{l} M_r(E_n) = Z_r(E_n)*S_r(E_n) = (m_{ij}) \\ m_{ij} = e_i \otimes e_j = z_{ij} \cdot (e_i o e_j) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \quad \text{Eq. (8)}$$

This matrix $M_r(E_n)$ defines the operation '⊗ over the elements of the set E_n.

Consider the matrices $Z_3(E_8)$ and $S_3(E_8)$ shown in Figure 2. If their star products, $Z_9(E_8)*S_3(E_8) = M_3(E_8)$ is formed, one finds that $M_3(E_8)$ has the same structure as $\mathfrak{M}_3(E_8)$. Moreover, if the submatrices $M_2(E_4) = Z_2(E_4)*S_2(E_4)$ and $M_1(E_2) = Z_1(E_2)*S_1(E_2)$ are similarly formed, one observes that they are also structurally similar to $\mathfrak{M}_2(E_4)$ and $\mathfrak{M}_1(E_2)$, respectively. This shows that all of the real division algebras C, H and O can be defined by ⊗-matrices of the type $M_r(E_n)$ defined by Eq. (8), where $n = 2^r$. Note that the Z-matrix $Z_3(E_8)$ shown in Figure 2(a) satisfies the following equations: $M_r(E_n)$ such that Eqs. (5), (6) and (7) hold. To do this, introduce two special matrices $Z_r(E_n)$ and $S_r(E_n)$ of the same dimensions n x n which are the *sign matrix* and *structure matrix*, respectively. The sign matrix

$$z_{ii} = -1 \text{ (if } i \geq 2)$$

$$z_{i1} = z_{1i} = +1 \text{ (for all i)} \qquad \text{Eq.(9)}$$

$$z_{ij} = -z_{ji} \text{ (if } i \neq j,\ i,j \geq 2)$$

which simply corresponds to the sign coefficients of Eq. (4). The structure matrix S_3 (E_8), on the other hand, defines the Klein group $<E_8\ ;o>$ of order n = 8. This group contains the Klein group $<E_4\ ;o>$ and $<E_2\ ;o>$ as subgroups which are defined by the submatrices $S_2(E_4)$ and $S_1(E_2)$, respectively. Thus, as noted earlier, the real division algebras O, H and C have a common *substratum:* the Klein group of order $n = 2^r$.

It is clear from the above discussions that the construction of the $\otimes$-matrix $M_r(E_n) = Z_r(E_n)*S_r(E_n)$ satisfying Eq. (4) can be carried out for any value of $n = 2^r$, where r is any positive integer. Such a matrix, in turn, can be used to construct a real division algebra A_r of order $n = 2^r$ which we shall call a **Cayley algebra of order** 2^r. In such an algebra any vector $a \neq O$ has a conjugate a* of the form given by Eq. (5), a norm N(a) given by Eq. (6) and an inverse a^{-1} given by Eq. (7).

THE ZSM PROCESS

To construct the $\otimes$-matrix $M_r \equiv M_r(E_n)$, first form a sign matrix $Z_r \equiv Z_r(E_n)$ that satisfies Eq. (9). Note, however, that there are many such sign matrices $Z_{r,k} \equiv Z_r(E_n)_k$ that satisfy these equations. Using Eq. (9), write: $Z_r = Z_{r(+)} + Z_{r(-)}$, where $Z_{r(+)}$ is symmetric while $Z_{r(-)}$ is skew. The skew matrix $Z_{r(-)} = (z^-_{ij})$ is such that $z^-_{ij} = -z^-_{ji}$ if $i \neq j$ and $i, j \geq 2$; otherwise $z^-_{ij} = 0$. Because of this the set $\mathcal{Z}_r$ of all Z-matrices $Z_{r,k}$ satisfying Eq. (9) has exactly $N(\mathcal{Z}_r) = 2^m$ distinct elements, where $m = \sum_{i=2}^{n-1} (n-1)$ and hence $k=1,\ldots,2^m$ With the aid of Eq. (8), these 2^m matrices $Z_{r,k} \in \mathcal{Z}_r$ can be used to construct 2^m $\otimes$-matrices of the form

$$M_{r,k} = Z_{r,k} * S_r \qquad (k = 1,\ldots,2^m), \qquad \text{Eq. (8.1)}$$

where $S_r = S_r(E_r)$ defines the Klein group $<E_n\ ;o>$ of order 2^r. These 2^m matrices $M_{r,k}$ form a set $\mathcal{M}_r$. Call this method of construction the ZSM Process.

Every $M_{r,k} \in \mathcal{M}_r$ defines a real division algebra $A_{r,k}$ of order $n = 2^r$. Hence, there are 2^m algebras of this type forming a set $\mathcal{A}_r$ which defines the class $\mathcal{C}[\mathcal{A}_r]$ of *Cayley algebras of order* 2^r. These 2^m algebras, however, are not all distinct. Since $M_{r,k} = Z_{r,k}*S_{r,k}$ defines $\otimes$ over E_n, then if P_π is an n X n permutation matrix associated with the permutation π on the n numerals $1,\ldots,n$ representing the n rows/columns of $Z_{r,k}$ it follows that the algebra $A^{(\pi)}_{r,k}$ defined by

$$M^{(\pi)}_{r,k} = (P_\pi Z_{r,k} P_\pi)*S_r = Z^{(\pi)}_{r,k} *S_r \qquad \text{Eq. (10)}$$

is isomorphic to the algebra $A_{r,k}$ defined by $M_{r,k}$, that is $A^{(\pi)}_{r,k} \cong A_{r,k}$. This isomorphism is determined by the one-to-one correspondence to the

$$\begin{array}{lccccccc} A_{r,k}: & 1 & 2 & 3 & \dots & i & \dots & n \\ & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \updownarrow \\ A^{(\pi)}_{r,k}: & \pi & \pi 2 & \pi 3 & \dots & \pi i & \dots & \pi n \end{array}$$

of the elements of their sets of basis vectors, where there are set $i = e_i$ and $\pi i = e_{\pi i}$ for simplicity. Although there are n! possible $n\times n$ permutation matrices P_π, only (n-2)! of these preserve the form of Z_r under the transformation: $Z_r \rightarrow Z^\pi_r = P_\pi Z_r P_\pi$. Thus, given any matrix $Z_{r,k} \in \mathcal{Z}_r$, there are also (n-2)! matrices $M^{(\pi)}_{r,k} \in \mathcal{M}_r$ that are *structurally equivalent* to $M_{r,k}$ and which define isomorphic algebras $A_{r,k}$. Hence, the Set $\mathcal{A}_r$ has at most $2^m/(n-2)!$ non-isomorphic (or distinct) Cayley algebras of order 2^r. Some of these algebras can also be obtained by the so-called *Cayley-Dickson Process* and are called *Cayley-Dickson Algebras*. The ZSM Process, on the other hand, enables one to obtain all of the 2^m members of the class $\mathcal{C}[\mathcal{A}_r]$ of Cayley algebras of order 2^r, where r is any positive integer. Thus, the following important

Theorem. There exists a class of Cayley algebras of every order 2^r, where r is any positive integer.

Every algebra $A_{r,k}$ in the class $\mathcal{C}[\mathcal{A}_r]$ contains a series of r-1 sub-algebras of orders $2^1, 2^2, \dots, 2^{r-1}$ which belong respectively to the classes $\mathcal{C}[\mathcal{A}_1], \mathcal{C}[\mathcal{A}_2], \dots, \mathcal{C}[\mathcal{A}_{r-1}]$. This means that $\mathcal{C}[\mathcal{A}_r]$ contains all of these smaller r-1 classes as subclasses in which each class $\mathcal{C}[\mathcal{A}_x]$ is contained in the next larger class $\mathcal{C}[\mathcal{A}_{x+1}]$. In general, since r is any positive integer, then there is an infinite number of classes which form an ascending series:

$$\mathcal{C}[\mathcal{A}_1] < \mathcal{C}[\mathcal{A}_2] < \dots < \mathcal{C}[\mathcal{A}_x] < \mathcal{C}[\mathcal{A}_{x+1}] < \dots < \mathcal{C}[\mathcal{A}_r] < \dots$$

This infinite series constitutes the Cayley family of real division algebras in which each class $\mathcal{C}[\mathcal{A}_x]$ determines a subfamily consisting of the finite ascending series: $\mathcal{C}[\mathcal{A}_1] < \mathcal{C}[\mathcal{A}_2] < \dots < \mathcal{C}[\mathcal{A}_x]$.

The class $\mathcal{C}[\mathcal{A}_1]$ contains only $2^0 = 1$ member A_1 which is isomorphic to the complex numbers C. $\mathcal{C}[\mathcal{A}_2]$ has $2^3 = 8$ members of which only four are nonisomorphic. On the other hand the class $\mathcal{C}[\mathcal{A}_3]$ has 2^{21} members all of which are nonassociative; at least 720 of these are isomorphic to the Cayley numbers O. Any algebra belonging to a class $\mathcal{C}[\mathcal{A}_r]$ in which $r \geq 2$ is noncommutative. And if $r \geq 3$, it is always nonassociative.

To illustrate the construction of Cayley algebras of order 2^r by the ZSM Process, consider the case of the 2^m algebras $A_{2,k}$ where $r = 2$ and $n = 4$. Here, $m = \sum_{i=2}^{3} (4\text{-}i) = 3$ and $N(Z_2) = 2^3 = 8$. Figure 3 shows the eight matrices $Z_{2,k}$ $(k=1,\ldots,8)$ which, together with the matrix S_2 shown in Figure 2(b), are used to form the matrices $M_{2,k}$ $(k=1,\ldots,8)$ shown in Figure 4. These matrices can be used to construct eight Cayley algebras $A_{2,k}$ of order $2^2 = 4$ forming the set $\mathcal{A}_2$ which defines the class $\mathcal{C}[\mathcal{A}_2]$. It can be shown that $A_{2,3} \cong A_{2,7}$, both of which are associative and $A_{2,1} \cong A_{2,5}$, $A_{2,2} \cong A_{2,6}$, $A_{2,4} \cong A_{2,8}$ all of which are nonassociative. The smallest nonassociative real division algebras are therefore of order $2^2 = 4$. Note that if the permutation matrix P_α represents the permutation $\alpha = (23)$ on the numerals 1234 representing the 4 rows/columns of $Z_{2,3}$, then $M^{(\alpha)}_{2,3} = (P_\alpha Z_{2,3} P_\alpha)^* S_2 = M_{2,7}$. Hence, $A_{2,3} \cong A_{2,7}$. In fact it can be shown that both $A_{2,3}$ and $A_{2,7}$ are isomorphic to the algebra Q of quaternions. Also, of the eight algebras in A_2, only $A_{2,3}$ and $A_{2,7}$ are associative and normed.

As a final example, Figure 5 shows the matrix $M_{4,p}$ which defines the Cayley algebra $A_{4,p}$ of order $n = 2^4 = 16$ belonging to the class $\mathcal{C}[\mathcal{A}_4]$. This is a real division algebra containing O, H and C. It is nonassociative and noncommutative, and it is not normed.

The Cayley algebras of order 2^r, where $r \geq 3$, are not just curiosities but they have important applications in both pure and applied mathematics. Thus, Eric Temple Bell remarked: "In passing, it seems rather remarkable that such a truncated algebra as [that of the Cayley numbers] could have any physical significance, but it has been applied to the quantum theory."

SUMMARY

This paper discussed real division algebras and showed that they have a common underlying structure exemplified by the algebra of Cayley numbers. This observation led to the construction of Cayley algebras of order 2^r, where r is any positive integer. In doing this, the ZSM Process was introduced using two special matrices (the sign matrix and structure matrix) to construct another matrix (the x-matrix) that defined the Cayley algebras of order 2^r. These algebras were shown to form a family of classes which established the existence of a class of Cayley algebras of every order 2^r where r is any positive integer.

$$
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mathfrak{M}_1 & 2 & -1 & 4 & -3 & 6 & -5 & -8 & 7 \\
 & 3 & -4 & -1 & 2 & 7 & 8 & -5 & 6 \\
\mathfrak{M}_2 & 4 & 3 & -2 & -1 & 8 & -7 & 6 & 5 \\
 & 5 & -6 & -7 & -8 & -1 & 2 & 3 & 4 \\
 & 6 & 5 & -8 & 7 & -2 & -1 & -4 & 3 \\
 & 7 & 8 & 5 & -6 & -3 & 4 & -1 & 2 \\
 & 8 & -7 & 6 & 5 & -4 & -3 & 2 & 1
\end{array}
$$

Figure 1. **The $\otimes$-matrix $\mathfrak{M}_3(E_8) = (m_{ij})$, where $m_{ij} = e_i \otimes e_j = z_{ij}$, which defines the real division algebra $\mathfrak{U}_3$ of order $2^3=8$ isomorphic to the *Cayley numbers***

$$
\begin{array}{c|cccccccc}
 & + & + & + & + & + & + & + & + \\
Z_1 & + & - & + & - & + & - & + & - \\
 & + & - & - & + & - & + & - & + \\
Z_2 & + & + & - & - & + & - & + & - \\
 & + & - & - & - & - & + & - & + \\
 & + & + & - & + & - & - & - & + \\
 & + & + & + & - & - & + & - & - \\
 & + & - & + & + & - & - & + & -
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
S_1 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
 & 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
S_2 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
 & 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
 & 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
$$

(a) $Z_3(E_8)$ (b) $S_3(E_8)$

Figure 2. **(a) $Z_3(E_8) = (z_{ij})$, i j = 1,...,8, is a special *sign matrix;* for simplicity, ± = ±1. (b) $S_3(E_8) = (e_{ij}$ i,j=1,...,8, where $e_{ij} = e_i o e_j$ is the *structure matrix* of the Klein group <E_8;o> or order 8; v=e_v.**

$$\underset{Z_{2,1}}{\begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & - & + \\ + & - & - & - \end{bmatrix}} \quad \underset{Z_{2,2}}{\begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & - & - \\ + & - & + & - \end{bmatrix}} \quad \underset{Z_{2,3}}{\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{bmatrix}} \quad \underset{Z_{2,4}}{\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & - \\ + & + & + & - \end{bmatrix}}$$

$$\underset{Z_{2,5}}{\begin{bmatrix} + & + & + & + \\ + & - & - & - \\ + & + & - & - \\ + & + & + & - \end{bmatrix}} \quad \underset{Z_{2,6}}{\begin{bmatrix} + & + & + & + \\ + & - & - & - \\ + & + & - & + \\ + & + & - & - \end{bmatrix}} \quad \underset{Z_{2,7}}{\begin{bmatrix} + & + & + & + \\ + & - & - & + \\ + & + & - & - \\ + & - & + & - \end{bmatrix}} \quad \underset{Z_{2,8}}{\begin{bmatrix} + & + & + & + \\ + & - & - & + \\ + & + & - & + \\ + & - & - & - \end{bmatrix}}$$

Figure 3. **Eight possible Z-matrices $Z_{2,k}$ that can be used to form eight ⊗-matrices $M_{2,k}$ (shown in Figure 4) satisfying Eq. (8)**

$$\underset{M_{2,1}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & -3 & -2 & -1 \end{bmatrix}} \quad \underset{M_{2,2}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 4 & 3 \\ 3 & -4 & -1 & -2 \\ 4 & -3 & 2 & -1 \end{bmatrix}} \quad \underset{M_{2,3}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 4 & -3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{bmatrix}} \quad \underset{M_{2,4}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 4 & -3 \\ 3 & -4 & -1 & -2 \\ 4 & 3 & 2 & -1 \end{bmatrix}}$$

$$\underset{M_{2,5}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -4 & -3 \\ 3 & 4 & -1 & -2 \\ 4 & 3 & 2 & -1 \end{bmatrix}} \quad \underset{M_{2,6}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -4 & -3 \\ 3 & 4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{bmatrix}} \quad \underset{M_{2,7}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -4 & 3 \\ 3 & 4 & -1 & -2 \\ 4 & -3 & 2 & -1 \end{bmatrix}} \quad \underset{M_{2,8}}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -4 & 3 \\ 3 & 4 & -1 & 2 \\ 4 & -3 & -2 & -1 \end{bmatrix}}$$

Figure 4. **Eight ⊗-matrices $M_{2,k}$ satisfying Eq. (4). Note that $M_{2,3}$ and $M_{2,7}$ are both isomorphic to the algebra Q of quaternions.**

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
M_1	2	-1	4	-3	6	-5	-8	7	10	-9	12	-11	14	-13	16	-15
	3	-4	-1	2	7	8	-5	-6	-11	12	-9	10	-15	16	-13	14
M_2	4	3	-2	-1	8	-7	6	-5	12	-11	10	-9	16	-15	14	-13
	5	-6	-7	-8	-1	2	3	4	-13	14	-15	16	-9	10	-11	12
	6	5	-8	7	-2	-1	-4	3	14	-13	16	-15	10	-9	12	-11
	7	8	5	-6	-3	4	-1	-2	-15	16	-13	14	-11	12	-9	10
M_3	8	-7	6	5	-4	-3	-2	-1	16	-15	14	-13	12	-11	10	-9
	9	-10	11	-12	13	-14	15	-16	-1	2	-3	4	-5	6	-7	8
	10	9	-12	11	-14	13	-16	15	-2	-1	4	-3	6	-5	8	-7
	11	-12	9	-10	15	-16	13	-14	3	-4	-1	2	-7	8	-5	6
	12	11	-10	9	-16	15	-14	13	-4	3	-2	-1	8	-7	6	-5
	13	-14	15	-16	9	-10	11	-12	5	-6	7	-8	-1	2	-3	4
	14	13	-16	15	-10	9	-12	11	-6	5	-8	7	-2	-1	4	-3
	15	-16	13	-14	11	-12	9	-10	7	-8	5	-6	3	-4	-1	2
	16	15	-14	13	-12	11	-10	9	-8	7	-6	5	-4	3	-2	-1

Figure 5. This ⊗-matrix $M_{4,p}$ defines a Cayley algebra of order n = 16. Note that its submatrices M_3, M_2 and M_1 have the same structures as those of O, H and C.

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