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SINGULAR GRAPHS: THE SUM OF TWO GRAPHS[1]

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ABSTRACT

The *sum* of two graphs G and H, denoted by $G + H$, is the graph obtained by taking disjoint copies of G and H and then adding every edge xy, where x is a vertex in G and y is a vertex in H. The sum $G + H$ of two graphs may be singular or non-singular, independently of the singularity or non-singularity of G and H.

Here, G and H are limited to complete graphs, paths and cycles. Formulas for det $A(G+H)$ are derived and consequently, necessary and sufficient conditions for the sum to be singular are obtained.

INTRODUCTION

By *graph* G is meant a pair $G = \{V(G), E(G)\}$, where $V(G)$ is a nonempty set of elements called *vertices*, and $E(G)$, is a set of 2-subsets of $V(G)$ called *edges*. The *adjacency matrix* of a graph G with vertices $v_1, v_2, \ldots, v_n$ is the n x n matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The graph G is said to be *singular* if $A(G)$ is singular, i.e., det $A(G) = 0$; otherwise, G is said to be *non-singular*.

The *sum* of two graphs G and H, denoted by $G + H$, is the graph obtained by taking disjoint copies of G and H and then adding every edge xy, where $x \in V(G)$ and $y \in H$.

The graph with n vertices where each vertex is adjacent to the remaining $n - 1$ vertices is called the complete graph of order n, denoted by K_n. The *path of order n*, denoted by P_n, is the graph with n distinct vertices $x_1, x_2, \ldots, x_n$ and whose edges are $x_i x_{i+1}$ for $i = 1, 2, \ldots, n-1$. The *cycle of length n*, denoted by C_n is the graph obtained from the path P_n by adding the edge $x_1 x_n$.

This paper studies the sum $G + H$ of two graphs G and H where G and H are any of the graphs K_p, P_q and C_r. Formulas for det $A(G + H)$ are derived and singular graphs $G + H$ are characterized.

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SUM OF TWO GRAPHS

Rara (1991) has computed for the determinants of $G + H$ where G and H are any of the graphs P_n, C_n, K_n with only three exceptions, namely, $C_m + C_n$, $C_m + K_n$ and $K_m + K_n$. The last sum is trivial since $K_m + K_n = K_{m+n}$. The problem on the first two sums is settled here.

The adjacency matrix of the sum of two cycles C_m and C_n has the following form:

$$A(C_m + C_n) = \begin{bmatrix} A(C_m) & M \\ N & A(C_n) \end{bmatrix}$$

where M and N are matrices of 1s having sizes m X n and n X m, respectively. For convenience, any matrix all of whose entries are equal to c will be called a *c-matrix*.

If A is any m X n matrix, denote by R_i its *ith* row and by Cj its *jth* column. The operation of interchanging rows (columns) u and v will be denoted by $R_u \leftrightarrow R_v$; $(C_u \leftrightarrow C_v)$. The operation of adding to R_i the linear combination of rows $c_1R_1 + c_2R_2 + \ldots + c_mR_m$ will be denoted by $[c_1R_1 + c_2R_2 + \ldots + c_mR_m] + R_i \rightarrow R_i$.

Theorem 1. If $m \equiv 0 \pmod 4$ *or* $n \equiv 0 \pmod 4$, *then* $C_m + C_n$ is singular.

Proof: Without loss of generality, assume that $m \equiv 0 \pmod 4$. Let $A = A(C_m + C_n)$ be the adjacency matrix of $(C_m + C_n)$. If the row operation $[-R_2 + R_4 - \ldots - R_{n-2}] + R_n \rightarrow R_n$ to A is applied, the *nth* row is clearly reduced to a zero row. Clearly, this row operation preserves the determinant of A. Thus, $C_m + C_n$ is singular. ■

Lemma 1. *Let n be odd and $A = A(C_n)$. Then A is transformed to the diagonal matrix* diag(1, 1,..., 1, 2) *by the following sequence of 5 determinant-preserving operations:*

(a) $R_1 \leftrightarrow R_2,\ R_2 \leftrightarrow R_3 \ldots,\ R_{n-1} \leftrightarrow R_n$

(b) $[-R_1 + R_3 + \ldots + (-1)^{\frac{n-1}{2}} R_{n-2}] + R_{n-1} \rightarrow R_{n-1}$

(c) $[-R_2 + R_4 + \ldots + (-1)^{\frac{n-1}{2}} R_{n-1}] + R_n \rightarrow R_n$

(d) $[-\frac{1}{2} R_n] + R_{n-1} \rightarrow R_{n-1},\ [-\frac{1}{2} R_n] + R_{n-2} \rightarrow R_{n-2}$

(e) $[-R_{n-1}] + R_{n-3} \rightarrow R_{n-3},\ [-R_{n-2}] + R_{n-4} \rightarrow R_{n-4}, \ldots, [-R_3] + R_1 \rightarrow R_1$

Proof. Verification is straightforward. ■

Lemma 2. *Let J be an n x r 1-matrix, where n is odd. Then the sequence of operations (a)-(e) transforms J to a matrix whose first n-*1 *rows form a* $\frac{1}{2}$ *- matrix and whose last row forms a 1-matrix.*

Proof. Verification is straightforward. ■

Lemma 3. *Let* $A = A(C_n)$, *where* $n \equiv 2 \pmod 4$. *Then A is transformed to the diagonal matrix diag*(1, 1, ..., 1, 2, 2) *by the following sequence of 5 operations:*

(a) $R_1 \leftrightarrow R_2, R_2 \leftrightarrow R_3 \ldots, R_{n-1} \leftrightarrow R_n$

(b) $[-R_1 + R_3 + \ldots - R_{n-2}] + R_{n-1} \rightarrow R_{n-1}$

(c) $[-R_2 + R_4 + \ldots - R_{n-1}] + R_n \rightarrow R_n$

(d) $[-\frac{1}{2} R_n] + R_{n-1} \rightarrow R_{n-1}, [-\frac{1}{2} R_n] + R_{n-2} \rightarrow R_{n-2}$

(e) $[- R_{n-1}] + R_{n-3} \rightarrow R_{n-3}, [- R_{n-2}] + R_{n-4} \rightarrow R_{n-4}, \ldots, [- R_3] + R_1 \rightarrow R_1$

Moreover, the above sequence of operations reverses the sign of det *A.*

Proof. Verification is straightforward. ■

Lemma 4. *Let J be an n x r 1-matrix, where n is odd. Then the sequence of operations (a)-(e) transforms J to a matrix whose first n-*2 *rows form a* $\frac{1}{2}$ *- matrix and whose last two rows form a 1-matrix.*

Proof. *Verification is straightforward.* ■

Theorem 2. *Let* $m \equiv 1$ *or* 3 (mod 4) *and* $n \equiv 1$ *or* 3 (mod 4). *Then* det $A(C_m + C_n) = 4 - mn$.

Proof. By applying (a)-(e) to the first m rows and to the last n rows of $A(C_m + C_n)$, the following matrix is obtained.

$$A' = \begin{bmatrix} I'_m & M \\ N' & I'_n \end{bmatrix} = \begin{array}{ccccc|ccccc} 1 & & & & & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & 1 & & & & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & & \ddots & & & \vdots & \vdots & & & \vdots \\ & & & 1 & & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & & & & 2 & 1 & 1 & \cdots & \cdots & 1 \\ \hline \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & 1 & & & & \\ \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & & 1 & & & \\ \vdots & \vdots & & & \vdots & & & \ddots & & \\ \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} & & & & 1 & \\ 1 & 1 & \cdots & \cdots & 1 & & & & & 2 \end{array}$$

where, by Lemmas 1 and 2, $I'_m = \text{diag}(1, 1, \ldots, 1, 2)$, M' has $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \ldots \frac{1}{2}]$ in the first $m - 1$ rows and $[11 \ldots 1]$ in the last row, $I'_n = \text{diag}\ (1, 1, \ldots, 1, 2,)$ N' has $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} \ldots \frac{1}{2}]$ in the first $n - 1$ rows and $[11 \ldots 1]$ in the last row. Multiply rows m and $m + n$ by $\frac{1}{2}$ to get the matrix

$$A'' = \begin{bmatrix} I''_m & M'' \\ N'' & I''_n \end{bmatrix}$$

where I_m, I_n are identity matrices and M'', N'' are matrices whose entries are all equal to $\frac{1}{2}$. If the operations $[-\frac{1}{2} R_1 - \frac{1}{2} R_2 - \ldots - \frac{1}{2} R_m] + R_i \rightarrow R_i$ for $i = m + 1, m + 2, \ldots, m + n$ is applied, the resulting matrix is block upper triangular and has the following form

$$A^* = \begin{bmatrix} I_m & M'' \\ O & I^*_n \end{bmatrix} = \begin{bmatrix} 1 & & & & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & 1 & & & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & & \ddots & & \vdots & \vdots & & & \vdots \\ & & & 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & & & & 1 \; \frac{1}{2} & \frac{1}{2} & \cdots & \cdots & \frac{1}{2} \\ & & & & p & q & q & \cdots & q \\ & & & & q & p & q & \cdots & q \\ & & & & \vdots & \vdots & \ddots & \cdots & \vdots \\ & & & & q & \cdots & q & p & q \\ & & & & q & \cdots & q & q & p \end{bmatrix}$$

where each entry in the diagonal of I^*_n is $p = 1 - \frac{m}{4}$ and each off-diagonal element is $-\frac{m}{4}$. It follows that $\det A^* = 1 - \frac{mn}{4}$ and consequently, $\det A(C_m + C_n) = 4 \det A^* = 4 - mn$. ■

Using a similar argument and Lemmas 3 and 4, the following theorem can be established:

Theorem 3. *Let* $m \equiv n \equiv 2 \pmod 4$. Then $\det A(C_m + C_n) = 16 - 4mn$.

By a similar argument and using Lemmas 1, 2, 3 and 4, the following theorem can be established:

Thereom 4. *Let* $m \equiv 1$ or $3 \pmod 4$ and $n \equiv 2 \pmod 4$. *Then* $\det A(C_m + C_n) = 2mn - 8$.

Theorems 1-4 may be summarized as follows: *The sum* $C_m + C_n$ is singular if and only if C_m or C_n is singular.

Consider now the sum $C_m + K_n$. In [4, 6, 7], it is shown that if x and y are two distinct vertices in a graph G with $N(x)\backslash y = N(y) \backslash x$, then $\det A(G) = -2 \det A(G \backslash x) - \det A(G) \backslash \{x, y\})$. Assume that $n \geq 2$ and apply this result to the graph $K_n + G$. Then by mathematical induction, get

$$\det A(K_n + G) = (-1)^{n+1}[n \det A(K_1 + G) + (n-1) \det A(G)]$$

Applying this to the graph $G = C_m$,

$$\det A(C_m + K_n) = (-1)^{n+1} [n \det A(K_1 + C_m) + (n-1) \det A(C_m)]$$

Now, $K_1 + C_m = W_m$, the wheel of order $m + 1$. Therefore,

$$\det A(C_m + K_n) = (-1)^{n+1} [n \det A(W_m) + (n-1) \det A(C_m)]$$

In [7], the following formulas are given:

$$\det A(W_m) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 4; \\ n-2, & n \equiv 1 \pmod 4; \\ 2, & n \equiv 2 \pmod 4; \\ -n & n \equiv 3 \pmod 4. \end{cases}$$

$$\det A(C_n) = \begin{cases} 0, & n \equiv 0 \pmod 4; \\ 2, & n \equiv 1 \pmod 4; \\ -4, & n \equiv 2 \pmod 4; \\ 2 & n \equiv 3 \pmod 4. \end{cases}$$

Therefore, the following theorem has been proved:

Theorem 5.

$$\det A(C_m + K_n) = \begin{cases} 0, & \text{if } m \equiv 0 \pmod 4; \\ (-1)^{n+1} (mn + 4n - 2), & \text{if } m \equiv 1 \pmod 4; \\ (-1)^{n+1} (6n - 4), & \text{if } m \equiv 2 \pmod 4; \\ (-1)^{n+1} (-mn + 2n - 2), & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

It is easy to check, based on the above result, that if $m \neq 0 \pmod 4$, then $\det A(C_m + K_n) \neq 0$. Thus, the following corollary.

Corollary. *$C_m + K_n$ is singular if and only if $m \equiv 0 \pmod 4$.*

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