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STRUCTURAL CHARACTERIZATION OF FINITE TOPOLOGICAL GRAPHS

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Introduction

A topological space gives rise to a graph in a very natural way. Let τ be a topology on a set X. Construct a graph G whose vertex-set is X, and where two distinct vertices x and y are adjacent if and only if $U \cap V \neq \phi$ for all $U, V \in \tau$ such that $x \in U$, $y \in V$. Equivalently, x and y are non-adjacent if and only if there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ but $U \cap V = \phi$. We shall call G, and every graph which can be constructed in this manner, a *topological graph*. We shall also say that the topology τ (or the topological space (X, τ) induces the graph G, and symbolically we shall write $\tau \rightarrow G$.

Example. Consider the topological space (X, τ), where $X = [1, 2, 3, 4, 5]$ and $\tau = [\phi, X, [1], [2], [1, 2], [1,4], [1, 2, 4]]$. The induced topological graph is shown in Fig. 1.1.

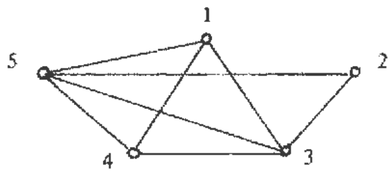

Fig. 1.1 A topological graph

Obviously, two homeomorphic topological spaces induce isomorphic graphs. However, non-homeomorphic topological spaces may induce isomorphic graphs. For example, the topology $\tau' = [\phi, X, [1], [2], [1, 2], [1, 4], [1, 2, 4], [1, 2, 3, 4]]$ on the same set X in the last example induces the same graph as τ does although (X, τ) and (X, τ') are non-homeomorphic.

For convenience, we shall adopt the following notations:

A	=	the closure of the subset A of a topological space X.
$[x, y]$	=	the edge of a graph with end vertices x and y.
P_4	=	a path with four vertices. This will also be denoted by $[a, b, c, d]$, where $[a, b]$, $[b, c]$, $[c, d]$ are distinct edges.
$d_G(x, y)$	=	the length of a shortest path in G whose end-vertices are x and y. Thus, $d_G(x, x) = 0$; $d_G(x, y) = 1$ if and only if x and y are adjacent; $d_G(x, y) = \infty$ if there is no path joining x and y.

$deg_G(x)$ = the number of edges in G containing the vertex x. This is called the *degree* of x in G.

Preliminary Results

The following two lemmas are easy and their proofs are omitted.

Lemma 2.1. Let β be a base for a topology τ on X and $\tau \to G$. Then two vertices x, y in G are adjacent if and only if $U \cap V \neq \phi$ for all U, $V \epsilon \beta$ such that $x \epsilon U, y \epsilon V$.

Lemma 2.2. Let τ, τ' be topologies on X and $\tau \to G, \tau' \to G'$. If τ is finer than τ' ($\tau \supseteq \tau'$), then G is a subgraph of G' ($G \subseteq G'$).

The next result is due to Diesto who is also doing some investigation on topological graphs.

Theorem 2.1. Let τ be a topology on X and $\tau \to G$. Then for each $x \epsilon X$, $\cap$ $[\overline{0} : 0 \epsilon \tau$ and $x \epsilon 0] \sim [x]$ is the set of all vertices adjacent to x.

Proof: Let y be a vertex adjacent to $x \epsilon X$. If $0 \epsilon \tau$ and $x \epsilon 0$, then $U \cap 0 \neq \phi$ for each $U \epsilon \tau$ that contains y. This implies that $y \epsilon \overline{0}$. Therefore, $y \epsilon \cap [\overline{0}: 0 \epsilon \tau$ and $x \epsilon 0] \sim [x]$.

Conversely, let $y \epsilon \cap [\overline{0}: 0 \epsilon \tau$ and $x \epsilon 0] \sim [x]$. Let $U, 0 \epsilon \tau$ such that $y \epsilon U, x \epsilon 0$. Since $y \epsilon \overline{0}$, it follows that $U \cap 0 \neq \phi$. Hence, y is adjacent to x.

Let $(X, \tau$ be a topological space and $\tau \to G$. For each $x \epsilon X$, let us define $S_G(x) = (y \epsilon X: y \neq x$ and y is not adjacent to $x]$. In view of theorem 2.1, this set is in τ since it is the complement of the closed set $\cap [\overline{0}: 0 \epsilon \tau$ and $x \epsilon 0]$. If A is a finite subset of X, then the set $S_G(A) = [x \epsilon X: x \notin A$ and x is not adjacent to any vertex in $A] = \widehat{x \epsilon A}\, S_G(x) \epsilon \tau$. Furthermore, if A and B are finite subsets of X, then $S_G(A) \cap S_G(B) = S_G(A \cup B)$. Thus, the sets $S_G(A)$, where A ranges over all finite subsets of X, form a base for some topology τ' on X. We shall call τ' the topology induced by the graph G. Observe that if G is any graph (not necessarily a topological graph), then it induces a topology with base consisting of the sets $S_G(A)$, where A ranges over all finite subsets of X, the vertex-set of G. If G induces the topology τ' we shall write symbolically $G \to \tau'$.

Theorem 2.2. If τ is a topology on X and $\tau \to G \to \tau' \to G'$, then $\tau \supseteq \tau'$ and $G \subseteq G'$.

Proof: We have already noted before that the sets $S_G(A) = [x \epsilon X: x \notin A$ and x is not adjacent to any vertex in $A]$, where A ranges over all finite subsets of X, are all in τ and that they form a base for τ'. Consequently, $\tau \supseteq \tau'$. By Lemma 2.2, $G \subseteq G'$.

Theorem 2.3. Let G be a finite graph and $G \to \tau \to G'$. Then $G' \subseteq G$.

Proof: Denote by X the vertex-set of G. We shall show that two vertices which are not adjacent in G must be non-adjacent in G'. Let x, $y \epsilon X$ be non-adjacent vertices in G; let $A = [v' \epsilon X: d_G(v, x \geqslant 2$ and $B = [v \epsilon X: d_G(v, y) \geqslant 2$. Then $S_G(A), S_G(B) \epsilon \tau$ and $x \epsilon S_G(A), y \epsilon S_G(B)$. We claim that $S_G(A) \cap S_G(B) = \phi$. Suppose that $z \epsilon S_G(A) \cap S_G(B)$. Then $z \notin A \cup B$ and z is not adjacent (in G)

to any vertex in $A \cup B$. It follows that $d_G(z, x) \leq 1$ and $d_G(z, y) \leq 1$. Now, z cannot be x or y since $d_G(x, y) \geqslant 2$. Therefore, $d_G(z, x) = d_G(z, y) = 1$. Since $y \in A$ and z is adjacent to y, then $z \notin S_G(A)$. This is a contradiction. Hence, $S_G(A) \cap S_G(B) = \phi$. This implies that x and y are not adjacent in G'.

The preceding theorem does not hold for infinite graphs. Consider a graph with an infinite number of connected components. If we denote this graph by G, and if $G \to \tau \to G'$, then it is easy to show that G' is complete, i.e., every pair of distinct vertices forms an edge in G'. This shows that G' properly contains G.

Combining Theorems 2.2 and 2.3, we get the following:

Theorem 2.4. Let G be a finite graph and $G \to \tau \to G'$. Then G is a topological graph if and only if $G = G'$.

Main Result

For convenience, we shall introduce the notion of a ***triangulator***. If e is an edge of a graph G, then any vertex x in G which is adjacent to both end-vertices of e shall be called a traingulator of e. The set of all triangulators of e in G shall be denoted by the symbol $T_G(e)$, or simply $T(e)$.

Theorem 3.1. A finite graph G is a topological graph if and only if for every subgraph $P_4 = [x_1, x_2, x_3, x_4]$ such that both end-vertices x_1 and x_4 are not triangulators of the middle edge $e = [x_2, x_3]$, there exists a triangulator v of e such that each vertex $u \notin e$ which is adjacent to v is itself a triangulator of e.

Proof: Let G be a finite topological graph and let $P_4 = [x_1, x_2, x_3, x_4]$ be a subgraph whose end-vertices do not belong to $T(e)$, where $e = [x_2, x_3]$. Let X denote the vertex-set of G and $A = [x \in X: d_G(x, x_2] \geqslant 2]$, $B = [x \in X: d_G(x, x_3) \geqslant 2]$. Observe that $x_4 \in A$ but $x_2 \notin A$. It is easy to see that $x_2 \in S_G(A)$. Similarly, $x_3 \in S_G(B)$. By Theorem 2.4, x_2 and x_3 are adjacent in G', where $G \to \tau \to G'$. Therefore, since $S_G(A), S_G(B) \in \tau$, it follows that $S_G(A) \cap S_G(B) \neq \phi$. Let $z \in S_G(A) \cap S_G(B)$. Then $z \notin A \cup B$ and z is not adjacent (in G) to any vertex in $A \cup B$. It follows that $d_G(z, x_2) = d_G(z, x_3) = 1$. Hence, $z \in T(e)$. In fact, we have shown that $\phi \neq S_G(A) \cap S_G(B) \subset T(e)$.

Now suppose that for all $v \in T(e)$, v is adjacent to some vertex $u \notin e \cup T(e)$. Consider again the sets A and B defined earlier. Take any $z \in S_G(A) \cap S_G(B)$. Then z is adjacent to some $u \notin e \cup T(e)$. We can assume, without loss of generality, that u is not adjacent to x_2. Therefore, $u \in A$. This is a contradiction since z is not adjacent to any vertex in A.

To prove the converse, let G be a finite graph with the property that for every subgraph $P_4 = [x_1, x_2, x_3, x_4]$ each of whose end-vertices is not a triangulator of the middle edge $e = [x_2, x_3]$, there exists a triangulator v of e such that every vertex u that is adjacent to v is in $e \cup T(e)$. Let $G \to \tau \to G'$. By Theorem 2.4, we need to show only that $G = G'$. By Theorem 2.3 we know that $G' \subseteq G$. Hence, it remains to prove that $G \subseteq G'$. Let x and y be adjacent vertices in G. We claim that these vertices are also adjacent in G'. If one end-vertex of the edge $[x, y]$ is of degree 1 in G, say $\deg_G(x) = 1$, then each $S_G(A) \in \tau$ containing y necessarily contains x. Thus, $[x, y]$ is an edge in G'. So let us assume that $\deg_G(x) > 1$, $\deg_G(y) > 1$ and consider the following cases:

Case 1. $[x, y]$ is not the middle edge of any subgraph P_4, both end-vertices of which are not triangulators of $[x, y]$.

In this case we can assume, without loss of generality, every vertex $v \neq y$ which is adjacent to x is a triangulator of $[x, y]$. Let A be a (finite) set of vertices in G such that $y \in S_G(A)$. We claim that $x \in S_G(A)$. Suppose that $x \notin S_G(A)$. Then x is adjacent to some vertex in A, say u. By assumption, u is a triangulator of $[x, y]$ and hence u is adjacent to y. This is a contradiction since $y \in S_G(A)$. Thus, $x \in S_G(A)$. It follows that x and y are adjacent in G'.

Case 2. $[x, y]$ is the middle edge of some subgraph $P_4 = [r, x, y, s]$ such that both r and s are not triangulators of $[x, y]$.

By assumption, there exists a triangulator v of $[x, y]$ such that every vertex u adjacent to v is in $e \cup T(e)$, where $e = [x, y]$. Let A, B be (finite) subsets of the vertex-set of G such that $x \in S_G(A)$, $y \in S_G(B)$. We claim that $v \in S_G(A)$. Suppose that $v \notin S_G(A)$. Then v is adjacent to some vertex $u \in A$. The vertex u cannot be x or y since $x \notin A$ and $y \notin A$. Therefore, $u \in T(e)$ and consequently, it is adjacent to both x and y. This is a contradiction since x is not adjacent to any vertex in A. Therefore, $v \in S_G(A)$. By a similar argument, we can show that $v \in S_G(B)$. Hence, $S_G(A) \cap S_G(B) \neq \phi$. It follows that x and y are adjacent in G'.

The following Corollaries are immediate consequences of Theorem 3.1:

Corollary 1. A finite graph G with girth $g \geqslant 4$ is not a topological graph.

Proof: If G is a finite graph with girth $g \geqslant 4$, then there exists a cycle $x_1, x_2, \ldots, x_g, x_1$ in G and this cycle has the shortest length. This cycle contains the path $P_4 = x_1, x_2, x_3, x_4$ and obviously x_1 and x_4 cannot be triangulators of the middle edge x_2, x_3. Moreover, x_2, x_3 does not have any triangulator since there are no cycles in G of length 3. Therefore, G is not a topological graph.

Corollary 2. Let G be a finite graph. If for every subgraph P_4, at least one of the end-vertices is a triangulator of the middle edge, then G is a topological graph.

Corollary 3. A finite and connected bipartite graph is a topological graph if and only if it is a star.

Proof: A bipartite graph does not contain odd cycles. Therefore, no edge of a bipartite graph can have a triangulator. Consequently, a finite and connected bipartite graph G is a topological graph if and only if it does not contain a subgraph P_4. Hence, G must be the complete bipartite graph $K_{1,n}$, i.e., a star.

Reference

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