# CYCLE GRAPHS 

Severino V. Gervacio<br>School of Graduate Studies<br>MSU - Iligan Institute of Technology<br>Iligan City, Philippines

For any given graph $G$, the cycle graph $C(G)$ has vertices which correspond to the chordless cycles of $G$, and two distinct vertices of $C(G)$ are adjacent if and only if the corresponding chordless cycles have at least one edge in common. The graph $G$ is cycle-vanishing if the iterated cycle graph $C^{n}(G)$ is acyclic for some finite integer $n$; otherwise $G$ is cycle-persistent. This paper gives a characterization of cycle-vanishing graphs, and in particular proves that $C^{3}(G)$ is acyclic if $G$ is cycle vanishing.

## Introduction

All graphs considered in this paper are finite, loopless and without multiple edges. Definitions of undefined terms and notions can be found in [3].

A chord in a cycle is an edge joining two non-consecutive vertices in the cycle.

Given a graph $G$, the associated cycle graph $C(G)$ has vertices which correspond to the chordless cycles of $G$, and two vertices of $C(G)$ are adjacent if and only if the corresponding chordless cycles of $G$ have at least one edge in common. In particular, if $G$ is acyclic then $C(G)$ is the null graph which we denote by $\phi$. For example, if $G$ is the graph shown in Figure 1(a), the chordless cycles of $G$ are $A_{1}=1231, A_{2}=236542, A_{3}=456874$, and $A_{4}=2368742$. Using $a_{i}$ to denote the vertex corresponding to the cycle $A_{i}$, the cycle graph $C(G)$ is shown in Figure 1 (b).

(b) Cl (G)


Figure 1. A graph $G$ and its cycle graph $C(G)$.

If $C^{n}(G)$ denotes the $n$-fold cycle graph of $G$, then $G$ is a cycle-vanishing graph if $C^{n}(G)=\phi$ for some positive integer $n$ and otherwise $G$ is a cycle-persistent graph. For example, if $G$ is the graph in Figure $1(a)$, then $C^{2}(G)=K_{2}$, the complete graph of order 2 , and $C^{3}(G)=\phi$; hence $G$ is cycle-vanishing. It is easily verified that $C\left(K_{4}\right)=K_{4}$, so the complete graph of order 4 is cycle-persistent. The problem of classifying cycle-vanishing graphs is the main concern of this paper.

The idea of cycle graph is motivated by the concept of line graph of a graph [5]. This is the graph $L(G)$ whose vertices correspond to the edges of $G$ and two vertices are adjacent if and only if the corresponding edges have a vertex in common. Perhaps the idea of clique graphs [4] was also motivated by the same concept.

## Some Basic Properties

If $H$ is any subgraph of $G$, the induced subgraph $\bar{H}$ is the subgraph with the same vertex set as $H$ and such that two vertices in $\bar{H}$ are adjacent if and only if they are adjacent in $G$. If $\bar{H}=H$, then $H$ is simply an induced subgraph of $G$.

## Lemma 1. If $C$ is any cycle in a graph $G$, then each edge of $C$ belongs to a chordless cycle of $G$ contained in the induced subgraph $\bar{C}$.

Proof. If $C$ is chordless, then $\bar{C}=C$ and the lemma holds. So suppose $C$ has chords. Then $C$ has order $n \geqslant 4$, and we may suppose the lemma holds for all cycles of smaller order in $G$. Any chord of $C$ divides $C$ into two cycles $C^{\prime}$ and $C^{\prime \prime}$ each of order less than $n$. Since all chords of $\bar{C}$ belong to $C$, then $\bar{C}$ ' and $\bar{C}$ ', are subgraphs of $\bar{C}$. The lemma therefore follows by induction on $n$.

Lemma 2. If $H$ is an induced subgraph of $G$, and $C$ is any cycle in $H$, then $C$ is chordless in $H$ if and only if it is chordless in $G$.
Proof. If $\bar{C}$ is the subgraph of $G$ induced by $C$ it is also the subgraph of $H$ induced by $C$, since $\bar{H}=H$. So the lemma follows.

Lemma 3. If $G$ has an induced subgraph which is cycle-persistent, then $G$ is cyclepersistent.
Proof. By Lemma 2, if $H$ is any induced subgraph of $G$ then $C(H)$ is an induced subgraph of $C(G)$. If $H$ is cycle-persistent, it follows that $C^{n}(H)$ is never the null graph, so $C^{n}(G)$ is never null, whence $G$ is cycle-persistent.
Theorem 1. If the graph $G$ contains an edge which belongs to at least four chordless cycles, then $G$ is cyle-persistent.

Proof. Let $H$ be a subgraph of $G$ containing four chordless cycles with a common edge. Then $\bar{H}$ contains the same four chordless cycles, and corresponding to them $C(\bar{H})$ contains a subgraph $K_{4}$. Since $K_{4}$ is cycle-persistent, it follows that $H$ is cyclepersistent, and hence $G$ is cycle-persistent, by Lemma 3.

A bridge of a graph $G$ is an edge which does not belong to any cycle in $G$. The bridge-free spanning subgraph of $G$, denoted by $\widetilde{G}$, is the graph obtained from $G$ by deleting all its bridges.

Lemma 4. $A$ graph $G$ is cycle-persistent if and only if its bridge-free spanning subgraph $G$ is cycle-persistent.
Proof. The cycles of $G$ are precisely the cycles of $\widetilde{G}$, so $C(G)=C(\widetilde{G})$, whence the lemma follows.

This lemma shows that in order for $G$ to be cycle-persistent, it is necessary and sufficient for some component of $G$ to be cycle-persistent. We can sharpen this result. Recall that a cut vertex of a graph $G$ is any vertex $v$ such that the graph $G-v$, induced by all vertices of $G$ except $\nu$, has more connected components than $(\dot{j}$ has. A block is a connected graph which has no cut vertices, and a block of a graph $G$ is a maximal connected subgraph which is a block. In a block of order at least 3 , any two vertices lie on a common cycle.
Theorem 2. Let $G$ be a connected bridgeless graph of order at least 3. Then $C(G)$ is connected if and only if $G$ is a block.
Proof. First assume that ( $\mathcal{B}$ is a block. Since its order is at least 3, it contains a cycle. If $G$ is a cycle, $C(G)=K_{1}$ so $C(G)$ is connected. Now suppose $G$ contains more than one cycle, and therefore at least two chordless cycles. Let $A, B$ be two chordless cycles, with corresponding vertices $a, b$ in $C(G)$. If $A, B$ have a common edge, then $a$ and $b$ are adjacent. If $A, B$ have no common edge they are joined by a path $P$. since $G$ is connected. Let us assume that $P$ has order at least $2(P$ may be of order 1 but this case is easier to handle). Let $u, v$ be the end vertices of $P$. with $u$ in $A$ and $v$ in $B$. Let $w$ be adjacent to $u$ in $A$, and let $x$ be adjacent to $u$ in $P$ ( x may coincide with v ). Since $G$ has no cutvertices, there is a path $Q$ with end vertices $w$ and $x$, which does not pass through $u$ (Figure 2). We may assume $Q$ has the least order among such paths. With the path $w u x$, the path $Q$ forms a cycle $C$. If $C$ has chords, the construction ensures that any chord it has must be incident with $u$. Thus, whether $C$ is chordless or not, there is a sequence of chordless cycles which begins with $A$ and ends with a cycle $C^{\prime}$ containing the edge $\iota x$, and consecutive cycles share a common edge (which happens to be incident with $u$ ). The subpath of $P$ which joins $C^{\prime}$ and $B$ has smaller order than $P$. Thus, iteration of the construc-


Figure 2. Construction for the sufficiency part of Theorem 2.
tion yields a sequence of chordlesss cycles $A=A_{1}, A_{2}, \ldots, A_{n}=B$ with the property that consecutive cycles have a common edge. Consequently,there is a corresponding path $a=a_{1}, a_{2}, \ldots, a_{n}=b$ in $C(G)$, whence $C(G)$ is connected.

Conversely, assume that $G$ is not a block. Since $G$ is connected and bridgeless each of its blocks has order at least 3 . Since $G$ is not a block, it has at least one cut vertex, say $x$. Theref ore $x$ is adjacent to vertices $u, v, w$ such that $u, v$ and $x$ lie on a chordless cycle $A$ and $w$ lies in a component of the subgraph induced by $G-x$ different from the one which contains $u$ and $v$ (Figure 3). The edge $x w$ lies in some chordless cycle $B$. Any sequence of chordless cycles $A=A_{1}, A_{2}, \ldots, A_{n}=B$ such that no two consecutive cycles are disjoint must be such that some pair shares only the vertex $x$. Thus, if $a$ and $b$ are the vertices of $C(G)$ corresponding to $A$ and $B$, there is no path between them, so $C(G)$ is disconnected.


Figure 3. Construction for the necessity part of Theorem 2.

Corollary. For any graph $G$, the connected components of $C(G)$ are the cycle graphs of the blocks of $G$ with at least 3 vertices.

## Characterization of Cycle-Vanishing Graphs

A cycle $C$ intercepts a tree $T$ if the intersection of $C$ and $T$ is precisely the set of end vertices of $T$. For convenience, suppose $C$ is a cycle embedded in the euclidean plane: two paths intercepted by $C$ are parallel if they can be drawn in the interior of $C$ so that they are internally disjoint (though they may have end vertices in common). Two paths intercepted by $C$ are skew if they are disjoint but are not parallel. These definitions are illustrated in Figure 4.


Figure 4. A family of paths each intercepted by the cycle $C$. The paths $P, Q S, T, U, V$ and $W$ are pairwise parallel; the paths $p$ and $Q R$ areskew; the paths $Q R$ and $Q S$ are neither parallel nor skew.

Lemma 5. A graph $G$ contains a cycle which intercepts two skew paths if and only if it contains a cycle which intercepts a tree with three end vertices.
Proof. This is simply a matter of two ways of describing the same configuration. Let $S, T, U$ be the three branches of a tree with three end vertices; let the paths $P, Q, R$ join the end vertices of the tree to form a cycle which intercepts the tree, so that $P$ joins the end vertices of $S T, Q$ joins the end vertices of $T U$, and $R$ joins the end vertices of $S U$ (Figure 5).


Figure 5. The configuration for Lemma 5.

Then $P, T, U$, and $R$ form a cycle which intercepts the skew paths $Q$ and $S$.
Let $C$ be a cycle which intercepts three parallel paths $P, Q, R$. Then $Q$ separates $P$ and $R$ (with respect to $C$ ) if $C$ contains paths $U$ and $V$ such that $Q$ joins $U$ and $V$, and PURV is a cycle which intercepts $Q$ (Figure 6). For example, note in Figure 4 that $T$ separates $P$ and $U$, but $U$ does not separate $T$ and $V$, relative to $C$.


Figure 6. Relative to $C$, the paths $P$ and $R$ are separated by the path $Q$.

Two paths intercepted by a cycle $C$ are $C$-independent if they are separated by some path intercepted by $C$; otherwise they are $C$-dependent. An ideal of $C$ is a maximal family of $C$-dependent paths. For example, in Figure 4 an ideal of $C$ is $\{\mathrm{T}, \mathrm{U}, \mathrm{V}, \mathrm{W}\}$.
Theorem 3. If $C$ is a cycle-vanishing graph, it contains no cycle which intercepts two skew paths.
Proof. We shall prove the contrapositive of the theorem. Let $C$ be a graph which contains a cycle that intercepts two skew paths. By Lemma $5, G$ contains a cycle $C$ which intercepts a tree $T$ with three end vertices $u, v, w$ (Figure 7).


Figure 7. A cycle $C$ intercepting a tree $T$ with three end vertices $u, v, w$.

In Figure 7, we show the cycles $X, Y, Z$ each being formed by a pair of branches of $T$ and a path in $C$. We can assume that $T$ has the least number of edges among all trees with three end vertices intercepted by some cycle in $G$. Furthermore, among all cycles intercepting $T$, we assume that $C$ has the shortest length. Then the subgraph $\bar{X}$ induced by $X$ does not contain edges like the ones indicated by the dashed lines in Figure 7. This condition also holds in the induced subgraphs $\bar{Y}$ and $\bar{Z}$. Hence, any chord in $X, Y$, or $Z$ is incident with the vertex $\rho$ common to the three branches of $T$. Therefore there is a sequence of chordless cycles $A_{1}$. $A_{2}, \ldots, A_{n}, A_{1}$ in $G$ such that consecutive cycles share a common edge (which happens to be incident with $p$ ). This sequence of chordless cycles in $G$ correspond to the cycle $a_{1} a_{2} \ldots a_{n} a_{1}$ in $C(G)$ where $a_{i}$ is the vertex corresponding to $A_{i}$. Let $e_{i}$ be an edge common to $C$ and $A_{i}(i=1,2,3)$. By Lemma $1, e_{i}$ lies in a chordless cycle, say $B_{i}$, contained in the induced subgraph $\bar{C}$. Clearly, $B_{i}$ is not equal to any cycle $A_{i}$. Let $b_{i}$ be the vertex in $C(G)$ corresponding to $B_{i}$. Then $a_{i}$ and $b_{i}$ are adjacent in $C(G)$ (Figure 8). Since $\bar{C}$ is a block, $C(\vec{C})$ is connected by Theorem 2.


Figure 8. A cycle $a_{1} a_{2} \ldots a_{n} a_{1}$ in $C(G)$ and three outside vertices $b_{1}, b_{2}, b_{3}$ adjacent to $a_{1}$, $a_{2}$ and $a_{3}$ respectively.

Therefore there exists a path $P$ in $C(\bar{C})$ joining $b_{1}$ and $b_{2}$. Also, there exists a path $Q$ in $C(\bar{C})$ joining $b_{3}$ to $P$ (Figure 8). The paths $P, Q$ and the edges $a_{i} b_{i}(i=$ $1,2,3)$ form a tree with three end vertices $a_{1} a_{2}, a_{3}$, intercepted by the cycle $a_{1} a_{2} \ldots a_{n} a_{1}$. Our argument actually shows that for each positive integer $n$, $C^{n}(G)$ contains a cycle (intercepting two skew paths). Hence $G$ is not cycle-vanishing.

A path of order at least two in a graph $G$ is reflexive if its end vertices are adjacent in $G$ otherwise it is irreflexive. Note that a path of order 2 is necessarily reflexive; an irreflexive path has order at least 3.

Theorem 4. Let $G$ be a graph composed of a cycle $C$ and a family $p$ of parallel intercepted paths. Then $G$ is cycle-vanishing if and only if it has the following properties:
(1) Any two irreflexive paths of $\rho$ are separated by at least two chords in $\rho$.
(2) Any ideal of $C$ consisting of at least 4 paths contains only reflexive paths.

Proof. First assume that $G$ is cycle-vanishing. Let $Q$ and $R$ be two irreflexive paths in $\rho$, and suppose they are separated by at most one chord in $\rho$. Let $U, V$ be the paths in $C$ forming a cycle $C^{\prime}$ with $Q$ and $R$. Then $C-C^{\prime}$ consists of two edgedisjoint paths $Q^{\prime}, R^{\prime}$ such that $A=Q Q^{\prime}$ and $B=R R^{\prime}$ are cycles (Figure 9). By


Figure 9. A cycle $C=Q^{\prime} U R^{\prime} V$ intercepting two irreflexive paths $Q . R$ separated by at most one chord $E$.

Lemma 1, there exists a chordless cycle $A_{1}$ containing $Q$ in the induced subgraph $\underline{A}$. Similarly, there exists a chordless cycle $B_{1}$ containing $R$ in the induced subgraph $\bar{B}$ (Figure 10).


Figure 10. An induced subgraph consisting of two chordless cycles $A_{1}, B_{1}$ and a cycle $Q U R V$ with at most one chord $E$.

In case $Q$ and $R$ are not separated by a chord, then the induced subgraph in Figure 10 has a cycle graph of order 6 and is shown in Figure 11 (a) if $U$ or $V$ has order at least 2, and Figure 11 (b) if both $U$ and $V$ have order 1 . In either case,

(a)

(b)

Figure 11. Cycle graph of an induced subgraph of $G$.
we have a cycle which intercepts two skew paths. It follows from Theorem 3 that $G$ is cycle-persistent. In case there is a chord $E$ which separates $Q$ and $R$, then $E$ lies in 4 chordless cycles. By Theorem 1, $G$ is cycle-persistent.

Let $K=\left\{Q_{1}, Q_{2}, \ldots, Q_{k}\right\}$ be an ideal of $C$ with $k 0 \geqslant$ paths. Let $R_{1}, R_{2}$, $\ldots, R_{k}$ be the edge-disjoint paths in $C$ whose end vertices are those of $Q_{1}, Q_{2}$. $\ldots, Q_{k}$ respectively and let $A_{i}(i=1,2, \ldots, k)$ be the cycle $Q_{i} R_{i}$ (Figure 12).


Figure 12. An ideal $K=\left\{Q_{1}, Q_{2}, \ldots, Q_{k}\right\}$ of $C$.
By Lemma 1, there exists a chordless cycle containing $Q_{i}$ in the induced subgraph $\overline{A_{i}}$. We may then assume, without loss of generality that each $A_{i}$ is a chordless cycle. Let the path $S_{i}$ be defined as $Q_{i}$ if $Q_{i}$ is irreflexive, and otherwise it is the edge joining the end vertices of $Q_{i}$.

Now suppose that there is an irreflexive path in $K$, say $Q_{i}$. Let $A$ be the cycle formed by $Q_{1}, S_{2}, S_{3}, \ldots, S_{k}$ and some paths in $C$; let $B$ be the cycle formed by $R_{1}, S_{2}, S_{3} \ldots, S_{k}$ and some paths in $C$. Then $A, B$ are chordless cycles having some common edges. Let $a, b$ be the vertices in $C(G)$ corresponding to $A, B$; let $a_{i}$
be the vertex in $C(G)$ corresponding to $A_{i}(i=1,2, \ldots, k)$. Then $a_{i}$ is adjacent to both $a$ and $b$ since the path $S_{i}$ is common to $A, B$ and $A_{i}$. Therefore we have at least 4 chordless cycles $a b a_{i} a(i=1,2, \ldots, k)$ in $C(G)$, each containing the edge $a b$. By Theorem $1, C(G)$ is cycle-persistent, and so $G$ is. This is a contradiction, and hence property (2) must necessarily hold.

We now prove the converse of the theorem by induction on the cardinality of $\rho$. It is easily verified when $|\rho| \leqslant 2$. Let $|\rho|=n \geqslant 3$ and assume that any graph consisting of a cycle and a family of less than $n$ parallel intercepted paths satisfying properties (1) and (2) is cycle-vanishing. Consider the following two cases:

Case 1. There are no chords in $\rho$. If each path in $\rho$ is reflexive then $C(G)$ is the complete bipartite graph $K_{1, n}$, which is acyclic. Theref ore $G$ is cycle-vanishing. If there is an irreflexive path in $\rho$, then by property (1), there is exactly one such ${ }_{3}$ path, say $S$ (Figure 13). Because of property (2) we must have $r \leqslant 2, s \leqslant 2$ and


Figure 13. A cycle $C$ intercepting reflexive paths $Q_{i} \cdot R_{i}$ and exactly one irreflexive path $C$.
hence $r+s \leqslant 4$. Since $n=r+s+1 \geqslant 3$, we also have $r+s \geqslant 2$. So $C(G)$ is one of the graphs in Figure 14, each of which is cycle-vanishing. Hence, $G$ is cycle-vanishing.


Figure 14. Cycle graph $C(G)$ for Case 1.
Case 2. There is a chord in $\rho$. Let $E$ be a chord in $\rho$. Split $G$ into subgraphs $G_{i}(i=1,2)$, each consisting of a cycle $D_{i}$ passing through $E$ and a family $\rho_{\mathrm{i}}$ of parallel intercepted paths (Figure 15). Properties (1) and (2) are inherited by each $\rho_{\mathrm{i}}$ from $\rho$. By induction hypothesis each $G_{i}$ is cycle-vanishing since $\left|\rho_{i}\right|<|\rho|$. By Lemma 1, each $G_{i}$ contains at least one chordless cycle passing through $E$. By Theorem 1 , each $G_{i}$ contains 2 chordless cycles passing through $E$. Furthermore, if one $G_{i}$ contains 2 chordless cycles through $E$, the other contains only 1 .


Figure 15. Splitting of $G$ along the chord $E$.

If $G_{1}, G_{2}$ each contains only one chordless cycle through $E$ then the cycle graph $C(G)$ is composed of $C\left(G_{1}\right), C\left(G_{2}\right)$ and one additional edge joining them. Therefore, $\left.C^{2}(G)=C^{2}\left(G_{1}\right) \cup C^{2}\right)\left(G_{2}\right)$, the union being disjoint. By induction hypothesis, it follows that $G$ is cycle-vanishing.

Suppose $G_{1}$ has 2 chordless cycles $A, B$ which pass through $E$. Then we may assume without loss of generality that $A$ contains an irreflexive path $Q$ of $\rho_{1}$. If $B$ contains any chord of $\rho_{1}$, at most one of them is in $A$, for $E, Q$ and any such chords form an ideal of $C$ and as $Q$ is irreflexive, property (2) limits the size of this ideal to at most 3. If $A \Delta B$ denotes the chordless cycle which is the symmetric difference of $A$ and $B$, it similarly follows that $A \Delta B$ contains at most two chords of $\rho_{1}$, for $Q$ together with such chords forms an ideal of $C$. Let $R$ represent a chord of $\rho_{1}$ in $A \triangle B$. Separate $G_{1}$ into subgraphs $H_{1}, H_{2}, H_{3}$ and $G_{0}$ such that $G_{0}$ is the subgraph containing $B$ and $Q$ while $H_{1}, H_{2}, H_{3}$ intersect $G_{0}$ in $R_{1}, R_{2}$ and $R_{3}$ respectively (Figure 16). In the general case, if any of the chords $R_{1}$ is absent, so is the related subgraph $H_{i}$.

Since $R_{i}$ is already contained in two chordless cycles in $G_{0}$, then there is a unique chordless cycle $Z_{i}$ in $H_{i}$ passing through $R_{i}$. Likewise $G_{2}$ contains a unique chordless cycle $X$ passing through $E$. Let $K$ be the subgraph of the cycle graph $C(G)$ induced by the vertices corresponding to $A, B, A \Delta B, X, Z_{1}, Z_{2}, Z_{3}$.


Figure 16. The graph G. $Q$ is an irreflexive path while $E$ is a chord.

Then the cycle graph $C(G)$ is shown in Figure 17. Clearly. $C^{2}(G)$ is the disjoint union of the five components $C^{2}\left(G_{2}\right), C^{2}\left(H_{1}\right), C^{2}\left(H_{2}\right), C^{2}\left(H_{3}\right)$ and $C(K)$. The first four are cycle-vanishing, by hypothesis. The last has only two chordless cycles so is cycle-vanishing. Thus $G$ is cycle-vanishing.


Figure 17. The cycle graph $C(G)$.

Corollary. Let $G$ be a cycle-vanishing graph. The each subgraph of $G$ consisting of a cycle and a maximal family of parallel intercepted paths satisfies properties (1) and (2) of Theorem 4.
Proof. Let $G_{0}$ be a subgraph of $G$ consisting of a cycle $C$ and a maximal family $\rho$ of parallel intercepted paths. If $x$ and $y$ are vertices in $G_{0}$ which are adjacent in $G$ but non-adjacent in $G_{0}$, then they must lie in only one path in $\rho$ and at most one of the vertices $x$ and $y$ is an end vertex of this path (Figure 18). We add the edge


Figure 18. A subgraph $G_{o}$ of $G$ consisting of a cycle $C$ and a maximal family of parallel intercepted paths.
$x y$ and remove the corresponding portion of the path forming a cycle with $x y$. We keep on repeating this process until all the paths in $\rho$ can no longer be shortened. Denote by $G_{1}$ the graph obtained from $G_{0}$ after completing the construction. Then $G_{1}$ is an induced subgraph of $G$ consisting of a cycle $C$ and a family $\rho$ ' of parallel intercepted paths which is in one-to-one correspondence with $\rho$. Furthernore, a path in $\rho^{\prime}$ is a chord if and only if the corresponding path in $\rho$ is a chord. Hence, it suffices to prove that $G_{1}$ satisfies properties (1) and (2). Since $G_{1}$ is an induced subgraph of $G, C_{1}$ is cycle-vanishing by Lemma 3 . By Theorem $4, G_{1}$ satisfies properties (1) and (2).

Theorem 5. If $G$ is cycle-vanishing graph, then each chordless cycle in the cyclegraph $C(G)$ has length 3.
Proof. Without loss of generality, we can assume that $G$ is a block. We shall prove the theorem by induction on the number $m$ of edges of $G$. The theorem is easily verified when $m \leqslant 5$. Let $m \geqslant 6$ and assume that the theorem holds when the graph has less than $m$ edges.

If $G$ does not contain any cycle with a chord then $G$ either is a cycle or is composed of a cycle intercepting exactly one irreflexive path. In both cases, $G$ is cycle-vanishing. So we may assume that $G$ contains a cycle $C$ intercepting a chord $E$. Let $\rho$ be the family of all paths intercepted by $C$. Observe that $C \cup \rho=G$ and that there are no skew paths intercepted by $C$ in view of Theorem 3. Split $G$ into subgraphs $G_{i}(i=1,2)$, each consisting of a cycle $D_{i}$ passing through $E$ and a family $\rho_{\mathrm{i}}$ of intercepted paths. By Lemma 1 , there is at least one chordless cycle in $G_{i}$ passing through $E$. By Theorem 1, each $G_{i}$ has at most 2 chordless cycles passing through $E$. Furthermore, if one $G_{i}$ has 2 chordless cycles passing through $E$, the other has only 1.

In case each $G_{i}$ has exactly one chordless cycle passing through $E$, then $C(G)$ is composed of $C\left(G_{1}\right), C\left(G_{2}\right)$ and an edge joining them. Therefore any chordless cycle of $C(G)$ either is in $C\left(G_{1}\right)$ or is in $C\left(G_{2}\right)$. By induction hypothesis, each chordless cycle in $C\left(G_{i}\right)$ or is in $C\left(\mathrm{G}_{2}\right)$. By induction hypothesis, each chordless cycle in $C\left(G_{i}\right)$ has length 3 . Hence, each chordless cycle in $C(G)$ has length 3.

In case $G_{1}$ has two chordless cycles $A, B$ which pass through $E$, then $G_{2}$ has only one chordless cycle $D$ passing through $E$. If $a, b, d$ are the vertices of $C(G)$ corresponding to $A, B, D$ respectively, then $C(G)$ consists of $C\left(G_{1}\right), C\left(G_{2}\right)$ and the cycle abda. By induction hypothesis, each chordless cycle in $C\left(G_{i}\right)$ has length 3 . Hence, each chordless cycle in $C(G)$ has length 3.
Corollary. If $G$ is a cycle-vanishing graph, then $C^{4}(G)=\phi$.
Proof. If $G$ is acyclic, $C(G)=\phi$. If $G$ contains cycles and $C(G)$ is acyclic, then $C^{2}(G)=\phi$. If $C(G)$ contains cycles, then all chordless cycles of $C(G)$ have length 3 .
By Theorem 1, any edge in $C(G)$ can lie in at most 3 chordless cycles. Therefore, $C^{2}(G)$ either is acyclic or it contains chordless cycles of length 3 which have no edges in common. In the first case, $C^{3}(G)=\phi$ and in the second case $C^{3}(G)$ is acyclic and so $C^{4}(G)=\phi$.

We can now give the main result of this section, which is a characterization of cycle-vanishing graphs.
Theorem 6. (Characterization Theorem). A graph G is cycle-vanishing if and only if it satisfies the following properties:
(1) $G$ does not contain a cycle intercepting two skew paths.
(2) $G$ does not contain an edge belonging to at least four chordless cycles.
(3) For every subgraph of $G$ which consists of a cycle and a maximal family $\rho$ of parallel intercepted paths, any two irreflexive paths in $\rho$ are separated by at least two chords in $\rho$ and any ideal of $C$ with at least four paths contains only reflexive paths.
Proof. First assume that $G$ is cycle-vanishing. Then (1) follows from Theorem 3, (2) follows from Theorem 1 and (3) follows from the Corollary to Theorem 4.

Conversely, let $G$ be a graph satisfying properties (1), (2) and (3.) Without loss of generality, we can assume that $G$ is a block of order at least 3. We shall prove that $G$ is cycle-vanishing by induction on the order $n$ of $G$. This is easily seen to be true when $n=3$ or 4 . Let $G$ be of order $n \geqslant 5$ and assume that the theorem holds for graphs of order less than $n$.

If $G$ does not contain any cycle with a chord then $G$ either is a cycle or is composed of a cycle intercepting exactly one irreflexive path. In both cases, $G$ is cycle-vanishing. So we may assume that $G$ contains a cycle $C$ intercepting a chord $E$. Let $\rho$ be the family of all paths intercepted by $C$. Then $G=C \cup \rho$ and there are no skew paths in $\rho$. Split $G$ into subgraphs $G_{i}(i=1,2)$, each consisting of a cycle $D_{i}$ passing through $E$ and a family pi of intercepted paths. Properties (1), (2), (3) are inherited by each $G_{i}$ from $G$ and hence $G_{i}$ is cycle-vanishing by hypothesis. Just like in the proof of Theorem 5, $G_{i}$ contains at least one and at most two chordless cycles passing through $E$ and if one $G_{i}$ contains two chordless cycles passing through $E$, the other one contains only one. By the same argument used in the proof of Theorem $4, G$ is cycle-vanishing.

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## References

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## Norman F. Quimpo, Discussant

Counting Cycles. The main result in this paper arises from a natural classification of cycle graphs suggested by the definition itself of a cycle graph. That is, if we derive a graph from the cycles of a graph, how many cycles are produced in the new graph? The answer is that the number may decrease, remain the same, or increase.

The paper deals with the first situation.
Regarding the second situation it will be interesting to characterize graphs which reproduce themselves in their cycle graphs. Wheels reproduce themselves.


## What other classes of graphs do?

An example of a graph whose cycle graph has more cycles than the original is the following:


How does the cycle count in the new graph relate to the configuration in the original? It seems like a big challenge to prove counting results here.

Verifying the Usual Properties. Now that a new way of defining a graph has been given, students of graph theory can have a heyday checking the usual properties of graphs. For example, what sort of graph produces a hamiltonian cycle graph? What conditions must be present in the original graphs for its cycle graph to be a tree, a bipartite graph, a regular graph? When does a graph have a graceful cycle graph? Etc., etc.

As more results about cycle graph pile up, and when the experts move in, we shall see the so-called "soft" results lead to "hard" ones. At some point, we can expect the key problems for cycle graphs arise.

Generalizations. The author imposes the restriction of chordless cycles in the definition of cycle graphs. Seeing the neat result that he obtained, we can see that it has been a good choice. The restriction is natural enough (a $K_{4}$ produces a $K_{4}$ ) strict enough to exclude a lot of graphs from the investigation and yet loose enough to yield a complex result.

However, we see the way to a generalized study. We can either (i) replace the chordless condition by another, or (ii) drop it entirely.

If we drop the chordless condition entirely, then the new cycle graph will have the same number of or more edges than the Gervacio cycle graph. It follows that any result on cycle-persistence will be preserved while any result on cycledisappearance will have to be reviewed. For example, consider the following graph under chordless or non-restricted conditions. Under the Gervacio conditon, it is cycle-vanishing. With no restriction, it becomes cycle-multiplying.


## Rolando E. Ramos, Discussant

Basically, Dr. Severino V. Gervacio's paper entitled "Cycle Graphs" is about cycles. Cycles are very important in graph theory. In fact, graph theory was developed from the problem of finding a cycle in a graph. Also, cycles have practical applications. One practical application is the so-called Chinese postman problem. From a post office, a postman goes from one block to another to deliver mails. Afterwards, he goes back to his office. If we represent the post office and the intersections by vertices, and the streets between them by edges, the postman's route turns out to be a cycle. Another application is the travelling salesman problem. Likewise, from a warehouse, a salesman travels from one town to another to sell his goods, then he goes back to his warehouse. Again, if we represent the warehouse and the towns by vertices, and the roads from one vertex to another by edges, the salesman's route becomes a cycle.

In this paper, Dr. Gervacio first defines two classes of graphs, namely, cyclevanishing graphs and cycle-persistent "graphs. Then he gives necessary and sufficient conditions for graphs to be cycle-vanishing or cycle-persistent. Most of the approaches in mathematics are of this type. A mathematician first defines a collection of objects then he finds objects that belong to the collection.

As pointed out by Dr. Gervacio, his study of cycle graphs was motivated by the concept of line-graphs. Line-graphs have theoretical applications and practical applications. They can be used in characterizing finite projective geometries, finite affine geometries and balanced incomplete block designs. They can be used also in solving coloring problems and self-avoiding walk problems. So, I am optimistic that mathematicians will be able to discover theoretical applications and practical applications of these cycle graphs.

## About the Authors

JULIAN ^. BANZON, Ph.D., Academician: Emeritus Professor of Food Science and Technology, University of the Philippines at Los Baños: Scientific Consultant of Maya Farms, Philippine Coconut Authority, and Philippine Coconut Research and Devclopment Foundation.

RODOLFO P. CABANGBANG, Ph. D., Associate Professor of Agronomy, University of the Philippincs at Los Baños; Exccutive Director, Cotton Research and Development Instjtute; one of the Outstanding Young Scientists of 1982.

BENJAMIN D. CAB RI:RA, M.D., M.P.H. (T.M.), Academician; Professor of Parasitology and Dean, Institute of Public Health, University of the Philippines.

PACIENTE A. CORIEERO, JR., D. Sc.; Regional Director, NSTA Region 8, one of the Outstanding Young Scientists of 1981.

AMANDO M. DALiSAY, Ph. D., Academician: former Consultant, Center for Policy and Development Studies. University of the Philippines at Los Baños.

JOSE ENCARNACIÓ , Jr.. Ph. D., Academician; Dean, School of Economics, Univer of the Philippines.

SEVERINO V. GER VACIO, Ph. D., Professor, Mindanao State University - Iligan Institute ot Technology, llizan City; one of the Outstanding Young Scientists of 1981.

RA A L. D. GUERRERO III, Ph. D.: National Team Leader for Agriculture, Fisheries Research Division, PC^RRD, Lo. Baños, Laguna, one of the Outstanding Young Scientists for 1983.

PO CIANO M. HALUS, Pl. D.; Product Development Manager, Monsanto Philippines Incorporat d, one of the Outstanding Young Scientists of 1983.

AL•JA: DRO N. HFRRIN, PlL D.; Director of Finance, School of Economics, University of the Philippines: As ociate Professor V, Conrado Benitez Associate Protessor of Denıographic l:conomics; one of the Outstanding Young Scientists of 1982.
l:E del MLNDO, M.D., Academician and National Scientist; Director and founder, The Childrens Medical Center Foundation of the Philippines: Director, Lungsod ng Kabataan; Professor Fmeritus, Far Eastern University.
M. ORFJANA, Ph. D.; Director, Institute of Fisheries Development and Research. University of the Philippines in the Visayas; one of the ()ut tanding Young Scientists of 1980.

LUZVISMINDA U. RIVIFRO. D. Sc.: Chairperson, Department of Chemistry, De La Salle University; Member, Kapisanan ng mga Kimiko sa Pilipinas: one of the Outstanding Young Scientists of 1983.

ERNESTO J. DEL ROSARIO, Ph. D.; Professor, Institute of Chemistry and National Institutes of Biochemistry and Applied Microbiology, University of the Philippines at Los Baños; one of the Outstanding Young Scientists of 1980.

JOVENTINO D. SORIANO, Ph. D., Academician; Professor of Botany, College of Science, University of the Philippines.

CLARA Y. LIM-SYLIANCO, Ph. D., Academician; Professor, Department of Chemistry, College of Science, and Professorial Lecturer, College of Medicine, both of the University of the Philippines.

GREGORIO T. VELASQUEZ, Ph. D., Academician and National Scientist; Emeritus Professor of Botany, University of the Philippines.

