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CONSISTENCY CONDITIONS FOR GROUP DECISIONS

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ABSTRACT

Some group decision rules that do not automatically satisfy Pareto optimality can be modified to do so by representing the alternatives in terms of lexicographically ordered vectors that depend on the feasible set. Certain conditions that have been considered as requirements for consistent group decisions might then be violated however. Less restrictive versions of those "consistency conditions" are proposed.

Introduction

A number of propositions in the literature on social choice -- the Chernoff (1954)-Sen (1977) Property α and Plott's (1973) path independence among them -- have been proposed as desiderata for consistent group decisions. In this paper we propose less restrictive versions of these "consistency conditions" that conform better to the logic of these requirements. We will describe an internally consistent group decision rule which violates the consistency conditions but satisfies the revised versions.

Preliminaries

The feasible alternatives constitute a nonnull set A in the set X of all possible alternatives, and a rule or procedure f is needed to determine a nonnull subset f(A) as the group's decision or choice set. Each person $k = 1, \ldots, n$ in the group is assumed to have a preference system U^k for evaluating alternatives $x, y, \ldots$ in X. From a knowledge of U^k one can infer whether or not $xP^k y$ (k prefers x to y). Putting $xR^k y$ if and only if $\neg yP^k x$, where $\neg$ denotes negation, R^k is k's ordering relation (reflexive, complete and transitive) on X; R_A^k is his ordering on A. Writing $\{U^k\} = (U^1, \ldots, U^n)$, $f(A)$ is short for $f(A, \{U^k\})$.

We will say that an ordering relation Q on X is nondictatorial if there is no k such that $Q_A = R_A^k$ for all A.

Assumption 1. There is a nondictatorial Q on X that is determined by $\{U^k\}$.

An example is provided by what Sen (1977) calls the "majority closure method," i.e. the transitive closure of the majority decision relation, which would have xQy if and only if there exist $z_1, \ldots, z_m \in X$ such that $x = z_1$, $y = z_m$, and for $i = 1, \ldots, m-1$, z_i gets at least as many votes in the group as z_{i+1} does in a pairwise comparison. The problem however is that this method allows a Pareto inferior alternative to belong to the choice set (Ferejohn and Grether 1977). As usual, x is Pareto inferior (or, simply, inferior) in A if there is a y in A such that yR_x^k for all k and yP^hx for some h, in which case y dominates x (or yDx). In order to rule out dominated alternatives in $f(A)$, we can employ a simple device.

Assumption 2. xP_Ay if and only if the first nonvanishing component of $r_A(x) - r_A(y)$ is positive, where $r_A(x) = (p_A(x), q(x))$, $p_A(x) = 1$ if x is undominated in A, 0 otherwise; and $q(x) \geqslant q(y)$ if and only if xQy.

The function q is either real-valued or vector-valued; in the latter case, $q(x) > q(y)$ means that the first nonvanishing component of $q(x) - q(y)$ is positive, i.e. the $q(x)$'s are ordered lexicographically. Putting xR_Ay if and only if $\sim yP_Ax$.

Assumption 3. $f(A) = \{x \in A \mid \forall y \in A: xR_Ay\}$.

Noting that not every x in A can be inferior if A is a closed set, which we assume, the lexicographic ordering of the alternatives by Assumption 2 assures a nonnull $f(A)$ containing no dominated points. In effect the procedure f in Assumption 3 would have two stages f_1 and f_2 whereby f_1 first selects undominated points after which f_2 then picks out the choice in accordance with Q. Specifically, define

$$A_1 = f_1(A) = \{x \in A \mid \forall y \in A: \sim yDx\}$$

$$A_2 = f_2(A_1) = \{x \in A_1 \mid \forall y \in A_1: xQ_Ay\}.$$

Then $f(A) = f_2(f_1(A)) = A_2$. Supplementing the majority closure method with the f_1 stage thus eliminates inferior alternatives from its $f(A)$. In the next section we describe a rule satisfying the Assumptions where the ordering by Q is itself lexicographic, but first we note some consequences for later use. (Lemma 1 is obvious; x is Q-greatest in a set B if xQy for all y in B.)

Lemma 1. If $A \subseteq B$, then if x is Q-greatest in B, x is Q-greatest in A.

Lemma 2. If $A \subseteq B$, then $A \cap B_1 \subseteq A_1$.

Since $A \cap B_1 = \{x \in A \cap B \mid \forall y \in B: \sim yDx\}$, and $(x \in A \cap B \;\&\; \forall y \in B: \sim yDx) \rightarrow (x \in A \;\&\; \forall y \in A: \sim yDx)$, if $A \subseteq B$, we get Lemma 2.

Lemma 3. If $x \in A_1 \cap B_2$ and $A_2 \subseteq B_1$, then $x \in A_2$.

Suppose the hypothesis is true, so that $x \in A_1 \cap B_1$ (because $B_2 \subseteq B_1$) and $\forall y \in B_1: xQy$. Since $A_2 \subseteq B_1$, x is Q-greatest in A_2 by Lemma 1. Hence $x \in A_2$ given that $x \in A_1$.

Lemma 4. $f_2(A_2 \cup B_2) \subseteq f_2(A_1 \cup B_1)$.

Assume $x \in f_2(A_2 \cup B_2)$. Then $x \in A_2 \cup B_2$ and $\forall y \in A_2 \cup B_2: xQy$. Since $A_2 \cup B_2 = \{y \in A_1 \mid \forall z \in A_1: yQz\} \cup \{y \in B_1 \mid \forall z \in B_1: yQz\}$, it follows that $x \in A_1 \cup B_1$ and $\forall z \in A_1 \cup B_1: xQz$.

Lemma 5. If $x \in A_1 \cup B_1$ and x is Q-greatest in $A_2 \cup B_2$, then $x \in A_2 \cup B_2$.

If the hypothesis is true, then xQy for all $y \in A_2 \cup B_2 = \{y \in A_1 | \forall z \in A_1 : yQz\} \cup \{y \in B_1 | \forall z \in B_1 : yQz\}$. So if $x \in A_1 \cup B_1$, we have $x \in A_2 \cup B_2$

A Group Decision Rule

Consider a committee whose members evaluate the alternatives in terms of the same set of criteria ranked in the same order of importance or priority. To each x corresponds a vector $u(x) = (u_1(x), u_2(x), \ldots)$ where u_i is a numerical function such that $u_i(x) > u_i(y)$ if x is better than y in terms of the ith criterion. While $u(x)$ is the same for all, different members may have different standards of acceptability with respect to any particular criterion: person k considers x acceptable as regards the ith criterion if and only if $u_i(x) \geqslant u_i^{k*}$. Writing $q_i^k(x) = \min\{u_i(x), u_i^{k*}\}$ and $q^k(x) = (q_1^k(x), q_2^k(x), \ldots)$, we assume that $xP^k y$ means $q^k(x) > q^k(y)$. Person k's ordering is thus determined by $u^{k*} = (u_1^{k*}, u_2^{k*}, \ldots)$, so that different members would have different orderings in general.

To obtain a decision rule under the Assumptions, we need only define $q(x) = (q_1(x), q_2(x), \ldots)$, where $q_i(x) = \min\{u_i(x), u_i^*\}$ and u_i^* is the median u_i^{k*}, assuming an odd number of members. The rationale for defining u_i^* as the median is that by Black's (1948) theorem on single-peaked preferences, it is the only choice for u_i^* that can win by simple majority rule over any other "candidate" for u_i^* (Encarnación 1969). Writing $\{u^{k*}\} = (u^{1*}, \ldots, u^{n*})$ and $u^* = (u_1^*, u_2^*, \ldots)$, $\{u^{k*}\}$ thus determines u^*, giving the ordering relation Q on X *in advance of any feasible set A*. Given A, $\{u^{k*}\}$ also determines $\{R_A^k\}$, which is needed only to eliminate inferior alternatives to get $A_1 = f_1(A)$, and Q_A then automatically yields $f_2(A_1) = f(A)$. Schematically,

$$\begin{array}{ccc} \{u^{k*}\} & \rightarrow & u \\ \downarrow & & \downarrow \\ \{R^k\} & & Q \\ \downarrow & & \downarrow \\ \{R_A^k\} & \rightarrow R_A & \rightarrow f(A) \end{array}$$

where $\{R_A^k\}$ and Q (through Q_A) determine R_A. The $\{U^k\}$ in Section 2 is given by $\{u^{k*}\}$ in the decision rule described, which we will denote by f^*. To be sure, f^* involves special assumptions, but we need only a counter-example to the consistency conditions. We only claim that f^* is internally consistent and not patently unreasonable, which will serve for the purpose.

It might be noted that f^* does not require an ordering relation R on X that suffices to yield R_A given A, contrary to Arrow's collective rationality condition which would have such an R in place of R_A in Assumption 3. Arrow's argument is that such a transitive R would make the group's decision independent of the particular sequence in which the feasible alternatives are presented for choice: "the basic problem is . . . the independence of the final choice from the path to it. Transitivity will insure this independence; from any [feasible set] there will be a chosen alternative" (Arrow 1963, p. 120). But clearly, transitivity on A and not necessarily on X would do for the purpose, as in Assumption 3. Collective rationality, which demands more than is really needed, is an unnecessary requirement.

Related to this point, Arrow's (1963, p. 26) argument for his independence of irrelevant alternatives (IIA) condition is that the group choice in A should be invariant with respect to changes in $\{R^k_{X-A}\}$, where X-A is the set of x's not in A. One could accept this requirement as reasonable, but Arrow's formalization of it makes R_A uniquely determined by $\{R^k_A\}$, which demands more than what was intended because of the Arrow format wherein R is uniquely determined by $\{R^k\}$. We observe that $f^*(A)$ depends only on R^k_A and Q_A and in no way varies with $\{R^k_{X-A}\}$. The f^* rule is thus in conformity with the motivation behind IIA but obviously fails it because $f^*(A)$ depends also on Q_A. In our view IIA, in requiring more than what is called for, is unnecessarily restrictive.

Consistency Conditions

Let g be a group decision rule, reserving the use of f for one that satisfies the Assumptions. Plott (1973) has pointed out that if path independence ("the independence of the final choice from the path to it" in the literal sense) is the objective of transitivity, one may even dispense with the latter if path independence can be had without it. He has proposed a formalization of this property.

Path independence (*PI*). If $A = B \cup C$, then $g(A) = g(g(B) \cup g(C))$.
Clearly $f(A)$ is path independent in the literal sense since R_A is transitive, yet it can violate *PI*. Since *PI* implies Property α, a violation of α will also show failure of *PI*.

Property α. If $A \subset B$, then $A \cap g(B) \subset g(A)$.
This condition has been considered as a "fundamental consistency requirement of choice" by many writers; see the references cited by Sen (1977, p. 67) and Kelly (1978, p. 26, n.2). But suppose f^* applies and consider Fig. 1 where the value of u_i^{k*} is indicated on the u_i axis by the numeral k. Person 1 has $u_1^{1*} = u_1(z)$, so he prefers z to x to y. Person 2 considers all three alternatives acceptable as regards u_1, and he prefers y to z to x on account of u_2. Person 3, who discriminates among the alternatives in regard to u_3, prefers z to y to x. The group's u^* is such that $q(x) > q(y) > q(z)$ because of u_4. Let $A = \{x, y\}$ and $B = \{x, y, z\}$. Then $f^*(A) = \{x\}$ but $f^*(B) = \{y\}$ because zDx, so that α is violated by $g = f^*$.

The idea behind the *PI* condition is that the choice in $A = B \cup C$ should come from the choices in B and C and should not depend on how A is disaggregated into

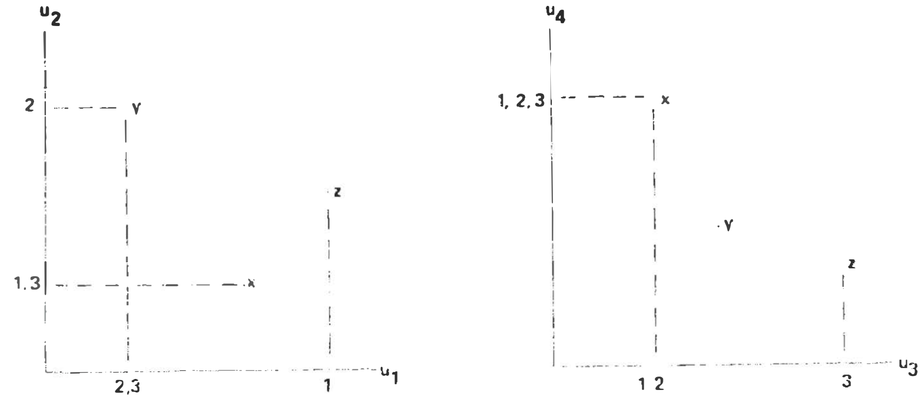

Figure 1

B and C. But this means that if the condition is to be reasonable, the choices in B and C should qualify as possible choices in the larger set A, which requires under the Assumptions that they be undominated in A. A revised version of *PI* follows from the Assumptions.

Theorem PI′. If $A = B \cup C$ & $f(B)) \subset A_1$ & $f(C) \subset A_1$, then $f(A) = f(f(B) \cup f(C))$.

Proof. Suppose the hypothesis is true. First we show that $f(A) \subset f(f(B) \cup f(C))$. Clearly $A_1 \subset B_1 \cup C_1$. Hence if $x \in f(A)$, then $x \in A_1$, x is Q-greatest in A_1 and $x \in B_1 \cup C_1$. Also, $B_2 \cup C_2 \subset A_1$, so that x is Q-greatest in $B_2 \cup C_2$ by Lemma 1. This with $x \in B_1 \cup C_1$ and Lemma 5 gives $x \in B_2 \cup C_2$, and therefore $x \in f(B_2 \cup C_2)$. To show $f(f(B) \cup f(C)) \subset f(A)$, suppose $x \in f(f(B) \cup f(C))$. This means $x \in B_2 \cup C_2$ and x is Q-greatest in $B_2 \cup C_2$. Since $B_2 \cup C_2 \subset A_1$, we have $x \in A_1$ so that $x \in A_2$ if x is Q-greatest in A_1. This is so, by Lemma 1, since $A_1 \subset B_1 \cup C_1$ and x is Q-greatest in $B_1 \cup C_1$, by Lemma 4, from the fact that $x \in B_2 \cup C_2$ and is Q-greatest in $B_2 \cup C_2$.

The rationale for α is that an alternative chosen in B, if still available when the feasible set has been reduced to A, should be among those chosen in A because it is "best" in the larger set and should therefore be best also in the smaller one. This would seem reasonable enough, but it implicitly assumes that the choices in A qualify as possible choices in B, which may not be the case. Making this assumption explicit,

Theorem α'. If $A \subset B$ & $f(A) \subset B_1$, then $A \cap f(B) \subset f(A)$.

Proof. Let the hypothesis be true. The conclusion is falsified if and only if there is an x such that $x \in A \cap B_2$ & $\sim x \in A_2$. Suppose such an x. Since $A \subset B$, Lemma 2 gives $A \cap B_1 \subset A_1$ so that $x \in A_1 \cap B_2$ since $x \in A \cap B_2$ and $B_1 \cap B_2 = B_2$. But $A_2 \subset B_1$ from the hypothesis, and therefore $x \in A_2$ by Lemma 3, contradicting $\sim x \in A_2$.

Four related conditions may be discussed together. Condition β + was introduced by Bordes (1976), β by Sen (1969), ϵ by Blair (as reported by Sen (1977, p. 69)) and δ by Sen (1971). Since β+ implies β, β implies ϵ, and ϵ implies δ, we need consider only β+ and δ.

Property β+. If $A \subset B$ & $A \cap g(B) \neq \phi$, then $g(A) \subset A \cap g(B)$.

Condition δ. If $x \in g(A)$ & $y \in g(A)$ & $A \subset B$, then ($\{x\} \neq g(B)$ & $\{y\} \neq g(B)$)

Suppose x is inferior in B and y is not. Then $\{x\} \neq f^*(B)$ but $\{y\} = f^*(B)$ is possible, showing failure of δ and the other conditions. These conditions put requirements on alternatives chosen in A when the feasible set is enlarged to B. As with *PI* and α, they fail to hold because of the possibility that an alternative chosen in a set may be dominated in a larger set. Restricting this possibility, the revised conditions become theorems as shown in the case of β+.

Theorem $\beta+'$. If $A \subset B$ & $f(A) \subset B_1$ & $A \cap f(B) \neq \phi$, then $f(A) \subset A \cap f(B)$.

Proof. Suppose there exists $y \in A \cap f(B)$, and suppose $x \in A \subset B$ and $x \in f(A)$. Then x is Q-greatest in A and therefore xQy since $y \in A$, so that $x \in f(B)$ since $y \in f(B)$ and x is undominated in B given the proviso that $f(A) \subset B_1$. Hence $x \in A \cap f(B)$.

Sen's (1971) Condition γ, which is equivalent to the following statement, is quite different from the others as it follows from the Assumptions.

Theorem γ. If $x \in f(A)$ & $x \in f(B)$, then $x \in f(A \cup B)$.

Under the hypothesis, x is in A_1 and in B_1 and Q-greatest in A_1 and in B_1, therefore $x \in A_1 \cup B_1$ and x is Q-greatest in $A_1 \cup B_1$, giving the conclusion. The reason for the difference is the fact that the hypothesis of γ does not allow x to be inferior in any of the sets considered. Plott's (1973) Axiom E, also called the Generalized Condorcet property by Blair *et al.* (1976), is a weaker version of γ and therefore also true.

There are other consistency conditions – Axioms 1 and 2 of Plott (1973) which are variations of the *PI* condition, α - of Sen (1977) and *B3* of Batra and Pattanaik (1972) which are weaker versions of α, and δ^* of Richelson (1978) which is a weaker version of δ – that are failed by f^*, but suitable reformulations are consequences of the Assumptions. In each case, the needed amendment is simply to make the alternatives chosen in smaller sets qualify as possible choices in some appropriate larger set.

Concluding Remarks

Noting that Properties α and β+ together are equivalent to Arrow's (1959) Definition C4 of a rational choice function, the consistency conditions are completely straightforward requirements on an individual's decision making: they are implied by the existence of a preference ordering (Arrow 1959, Theorem 2). They are however less compelling for group choices because of features in the latter which are absent for an individual, in particular, Pareto properties of alternatives depend on the feasible set. While it is quite correct to say that one can infer an individual's choices over larger sets from his choices over two-element sets, this may not hold for the group, for if the Pareto property of an alternative is considered important, group evaluation of an alternative may vary with the feasible set. Using the f^* rule defined in Section 3 we have shown possible conflicts between Pareto optimality and most of the consistency conditions (the α and $\beta+$ classes in contrast to the γ class), accordingly we would propose revised conditions which interestingly enough are consequences of the Assumptions in Section 2. Group decision rules that satisfy the consistency conditions obviously satisfy the proposed revisions, so the latter are less restrictive.

It might be argued that instead of abandoning Property α, which seems generally considered to be fundamental, and replacing it with α' one could equally well retain α and reject f^* as "unreasonable." (Indeed that was the reaction of a reader of an earlier draft of this paper.) But f^* is internally consistent while α is supposed to be a necessary condition for consistency. Given any feasible set A, $f^*(A)$ is derived from a relation which is nondictatorial and transitive on A, hence path independent; moreover, $f^*(A)$ is Pareto optimal and independent of alternatives outside A. Such properties would seem to be both necessary and sufficient for a valid counter-example to α. Also, the only difference between α and α' is that the latter makes explicit what must be implicitly assumed by α if Pareto optimality is necessary, viz. that the choices in the smaller set qualify as possible choices in the larger set. Clearly we have a conflict here between Pareto optimality and α which is resolved by α', and we would conclude that α' is the more reasonable requirement.

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Fr. Bienvenido F. Nebres, S.J., Discussant

I used to be hesitant about discussing applications of mathematical logic to various sciences. The reason is that these applications seem so esoteric. However, in recent years we have seen the languages developed in mathematical logic having more and more importance in various aspects of computer science. It seems to me that present efforts at formalization of aspects of the social sciences can help in computer modeling of social science-type situations.

I would like to thank Dr. Encarnación for a very interesting and well-written paper. My only regret is not having familiarity with the earlier literature on the subject. Thus I cannot properly weigh the arguments in favor of the modification α' of property α. However, I find the presentation clear – in particular, breaking up the steps in the construction of the choice function so as to explicitate the elimination of dominated points first. This then clarifies the introduction of the additional condition

$$f(B) \subset A_1, (f(A) \subset B_1) \text{ in Theorem } PI' \text{ (Theorem } \alpha'\text{)}.$$

What I found more intriguing from a mathematical point of view was the "unreasonable" function f^*. According to Dr. Encarnación, an earlier reader suggested getting rid of functions such as f^*. Mathematical experience, however, shows that such unreasonable functions often hold the key to a deeper understanding of a theory. For example, that most unreasonable function, the Dirac δ - function, arising from physics opened the way to the theory of generalized functions (distributions) and a new era in analysis and partial differential equations. I would just like to indicate one question which f^* brings up:

Person k's ordering is given by $q^k = (q_1^k, q_2^k, q_3^k)$ and on calculation:

$$q^1(z) > q^1(x) > q^1(y)$$

$$q^2(y) > q^2(z) > q^2(x)$$

$$q^3(z) > q^3(y) > q^3(x)$$

A "naive" look at the orderings of choices would indicate that z should be first, y second, x third. That is, the q function should give: $q(z) > q(y) > q(x)$.

As a matter of fact, the q-ordering is the reverse: $q(x) > q(y) > q(z)$. What happens? The definition of q is dependent on the q_i and on the medians u_1^* of u_i^{1*}, u_i^{2*}, u_i^{3*}, the criteria for acceptability. In the case of the choice function f^*, the medians u_1^* dampen the contribution of the q_i's.

Of course, the question raised by this discrepancy between the individual choice functions q^k and the decision rule q for the group is no longer one of consistency, which is the concern of the paper. It is a question of *correctness* of formalization. One of the central questions in theoretical computer programming today is correctness of programs: How do I know that the program I wrote does what I intended it to do? Similarly, in our case, the discrepancy between the q^k and q raises a question of correctness of formalization: Does the construction of the q^k and q do what we meant them to do, i.e., give a mathematical formalization for informal decision processes? More precisely, are there other ways of constructing q^k, q (for example, using a different u_i^* than the median) which avoids seeming violations of intuition such as comes up in the function f^*?

In any case, I would like to thank Dr. Encarnación once more for this paper. I hope it stimulates greater interest in the challenge of developing mathematical formalisms in the social sciences