Trans. Nat. Acad. Sci. & Tech. (Phils.)
1985.7:269-274

n-cycle Block Design Graphs

Severino V. Gervacio
Iligan Institute of Technology
Mindanao State University
Iligan City, Philippines

ABSTRACT

In 1976, K.M. Koh and Y.S. Ho introduced and initiated the study of a class of graphs which they called n-BD graphs (BD stands for block design). If the largest complete subgraph of a graph G has order n and if there exist positive integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that each i-complete subgraph of G is contained in exactly λ_i distinct n-complete subgraphs of G, then G is called an *n-BD graph*.

The author, in the same year, 1976, introduced and studied a class of graphs having some similarity in structure to the n-BD graphs. If G is a graph whose longest cycle is of length n and if there exist positive integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that each i-path in G lies in exactly λ_i distinct n-cycles of G, that G is called an *n-cycle BD graph.*

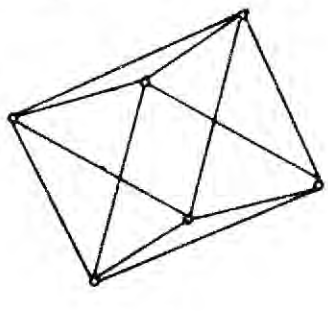
In this paper we characterize n-cycle BD graphs. Specifically, we show that the cycles of length at least 3, the complete graphs of order at least 3 and the complete 2-equipartite graphs of order at least 4 comprise all the n-cycle BD graphs.

Introduction

In this paper, by a *graph* we shall understand a finite undirected graph with no loops nor multiple edges. We shall use the symbol $G = \langle V(G), E(G) \rangle$ to denote a graph G with vertex-set $V(G)$ and edge-set $E(G)$.

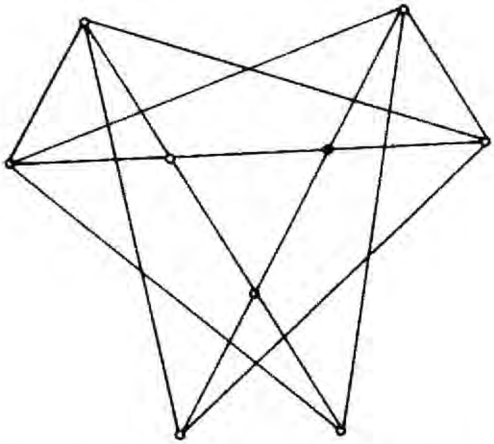
In 1976, K. M. Koh and Y. S. Ho [3] introduced and initiated the study of *n-BD graphs* (BD stands for Block Design). A connected graph G is called an n-BD graph if the maximum clique in G is K_n and there exist positive integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that each K_i in G is contained in exactly λ_i copies of K_n $(i = 1, 2, \ldots, n)$. The constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the *parameters* of G.

Example 1. The following graph is a 3-BD graph with parameters $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 1$.



We note here that the parameters λ_1, λ_2, λ_3 form a geometric sequence. Koh and Ho [4] have shown that the only n-BD graphs whose parameters form a geometric sequence are the n-equipartite graphs.

Example 2. The following graph is a 3-BD graph with parameters $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$.



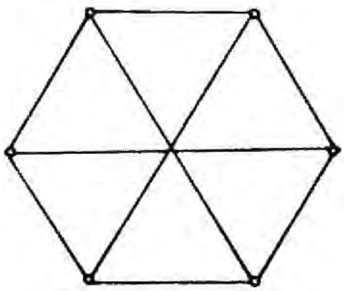
The graph in this example belongs to a class of n-BD graphs associated with the sequence of parameters $\lambda_1 = k$, $\lambda_2 = \ldots = \lambda_n = 1$. These graphs are studied by Koh and Ho [5].

In this paper, we shall deal with a class of graphs having some similarity in structure to n-BD graphs.

n-Cycle BD Graphs

Let G be a connected graph such that the maximum length of a cycle in G is n. If there exist positive integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that each path P_i in G is contained in exactly λ_i copies of an n-cycle C_n ($i = 1, 2, \ldots, n$), then G is called an *n-cycle BD graph.* The constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the parameters of G.

Example 3. The following graph is a 6-cycle BD graph with parameters $\lambda_1 = 6, \lambda_2 = 4, \lambda_3 = 2, \lambda_4 = 1, \lambda_5 = 1, \lambda_6 = 1$.



It is interesting to note that the graph in this example is at the same time a 2-BD graph with parameters $\lambda_1 = 3$, $\lambda_2 = 1$.

THEOREM 1 If G is an n-cycle BD graph, then its parameters satisfy the inequalities $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 1$.

Proof. That each $\lambda_i \geqslant 1$ follows from the definition of an n-cycle BD graph. We claim that if $1 \leqslant i < n$, then $\lambda_i \geqslant \lambda_{i+1}$. Consider a path $P_{i+1} = [v_1, v_2, \ldots, v_{i+1}]$. This path is contained in exactly λ_{i+1} copies of C_n. Therefore the path $P_i = [v_1, v_2, \ldots, v_i]$ is contained in at least λ_{i+1} copies of C_n. Hence, $\lambda_i \geqslant \lambda_{i+1}$.

THEOREM 2 Let G be an n-cycle BD graph with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let G contain exactly λ_n copies of C_n. Then

(a) $\lambda_n = |V(G)|\lambda_1/n$, and

(b) for $1 \leqslant i \leqslant j \leqslant n$, each path P_i is contained in exactly $(j - i + 1)\lambda_i/\lambda_j$ paths P_j.

Proof. (a) Each vertex in G is contained in exactly λ_1 copies of C_n. Hence, $|V(G)|\lambda_1$ counts all the n-cycles in G. However, each C_n is counted exactly n times since it contains exactly n vertices. Hence, the total number of n-cycles in G is $|V(G)|\lambda_1/n$.

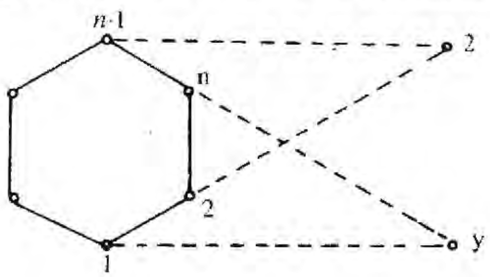
(b) Let $1 \leqslant i \leqslant j \leqslant n$ and denote by k the number of paths P_j containing a given path P_i. Then $k\lambda_j$ counts all the n-cycles containing P_i. Now, each cycle C_n is clearly counted exactly $j - i + 1$ times in the expression $k\lambda_j$. Hence, $\lambda_i = k\lambda_j/(j - i + 1)$, or $k = (j - i + 1)\lambda_i/\lambda_2$.

COROLLARY. An n-cycle BD graph with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ is regular of valency $2\lambda_1/\lambda_2$.

THEOREM 3. If G is an n-cycle BD graph, then $\lambda_n = \lambda_{n-1} = 1$.

Proof. Consider any path P_n, say $[1, 2, \ldots, n]$. Since $\lambda_n \geqslant 1$, P_n must lie in some n-cycle. Hence, n and 1 are necessarily adjacent. It follows that P_n lies in a unique n-cycle, namely $[1, 2, \ldots, n, 1]$ and so $\lambda_n = 1$.

Consider any n-cycle in G, say $[1, 2, \ldots, n, 1]$. This contains the path $[1, 2, \ldots, n\text{-}1]$ with $n - 1$ vertices. We claim that no other n-cycle contains this path. Suppose another n-cycle, say $[1, 2, \ldots, n - 1, x, 1]$, contains the path. Thus,



$x \neq 1, 2, \ldots, n$ and $\lambda_{n-1} \geq 2$. It follows that the path $[2, 3, \ldots, n]$ which also has $n-1$ vertices is contained in some other n-cycle $[2, 3, \ldots, n, y, 2]$, where $y \neq 1, 2, \ldots, n$. If $x = y$, then we get the cycle $[1, 2, \ldots, n, x, 1]$ which is of length $n+1$. If $x \neq y$, then we get the cycle $[1, n, y, 2, 3, \ldots, n-1, x, 1]$ of length $n+2$. In both cases we have a contradiction since n is the maximum length of a cycle in G. Hence, $\lambda_{n-1} = 1$.

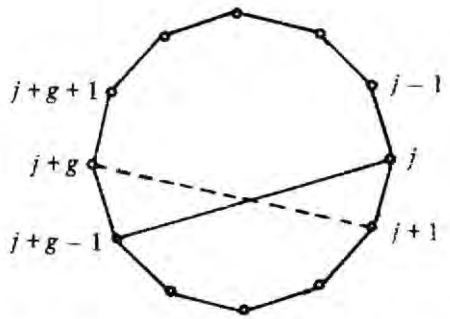
THEOREM 4. An n-cycle BD graph is hamiltonian.

Proof. Let $G = \langle V(G), E(G) \rangle$ be an n-cycle BD graph and let $C_n = [1, 2, \ldots, n, 1]$ be an n-cycle in G. We claim that C_n is a hamiltonian cycle in G. Suppose that C_n is not a hamiltonian cycle in G. Then there exists a vertex $x \in V(G), x \neq 1, 2, \ldots, n$. Since G is connected, we can assume without loss of generality that $[1, x] \in E(G)$. The path $[x, 1, n, n-1, \ldots, 3]$ which contains n vertices must lie in exactly one n-cycle. Hence $[x, 3] \in E(G)$. But then the path $[3, 4, \ldots, n, 1]$ would lie in the n-cycles C_n and $x, 3, 4, \ldots, n, 1, x]$. This contradicts the fact that $\lambda_{n-1} = 1$. Hence, G must be hamiltonian with C_n as one hamiltonian cycle.

Remark. Theorem 4 together with Theorem 2 (a) tell us that the total number of n-cycles in an n-cycle BD graph is λ_1.

LEMMA. Let $[1, 2, \ldots, g\ 1]$ and $[1, 2, \ldots, n, 1]$ be g, and n-cycles respectively in an n-cycle BD graph $G = \langle V(G), E(G) \rangle$ whose girth g is less than n. Then $[j, j+g-1] \in E(G)$ for $j = 1, 2, \ldots, n$.

Proof. We shall prove our Lemma by induction on j. The Lemma is obviously true for $j = 1$ since $[1, g] \in E(G)$. Assume that $[j, j+g-1] \in E(G)$, where $1 < j < n$. Consider the path $[j+g, j+g+1, \ldots, n, 1, 2, \ldots, j, j+g-1]$. This path has length



$n-g+2 < n$ and must therefore be contained in some n-cycle. Since $1, 2, \ldots, n$ are all the vertices in G, then $j+g$ must be adjacent to one of the vertices $j+1, j+2, \ldots, j+g-2$. Since g is the minimum length of a cycle in G then $j+g$ can only be adjacent to $j+1$, i.e., $[j+1, j+g] \in E(G)$. This completes our proof by induction.

THEOREM 5. Let G be an n-cycle BD graph. Then G has girth 3 or 4 or n.

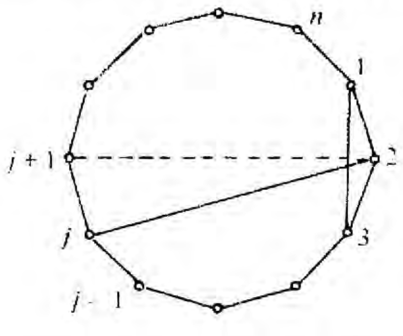
Proof. Let G be an n-cycle BD graph with girth g. If $g = n$, then we're done, if $g < n$, let $[1, 2, \ldots, g, 1]$ and $[1, 2, \ldots, n, 1]$ be g- and n-cycles respectively in G. According to the preceding Lemma, $[j, j + g - 1] \in E(G)$ for $j = 1, 2, \ldots, n$. In particular, $[2, g + 1] \in E(G)$. Hence $[1, 2, g + 1, g, 1]$ is a 4-cycle in G. It follows that $g = 3$ or 4.

We are now ready to state and prove our main result which characterizes all n-cycle BD graphs.

THEOREM 6. A graph G is an n-cycle BD graph if and only if either G is a cycle C_n $(n \geq 3)$, or G is a complete graph K_n $(n \geq 3)$, or G is a complete bipartite graph $K_{m,m}$ with $n = 2m$, $m \geq 2$.

Proof. The proof of sufficiency is easy and straightforward. To prove the necessity, let G be an n-cycle BD graph. If g is the girth of G, then either g is 3 or 4 or n. If $g = n$, then G is a cycle C_n. If $g < n$, then $g = 3$ or 4. Let us consider the following two cases.

Case 1. $g = 3 < n$. Let $[1, 2, 3, 1]$ and $[1, 2, \ldots, n, 1]$ be 3-and n-cycles respectively in G. We claim that the vertex 2 is adjacent to the vertices $3, 4, \ldots, n$. Clearly, 2 is adjacent to 3. Assume that 2 is adjacent to j, where $3 < j < n$. Consider the path $[j, j - 1, \ldots, 4, 3, 1, n, n - 1, \ldots, j + 1]$. This is a path with $n - 1$ vertices



and must therefore be contained in exactly one n-cycle. It follows that $j + 1$ is adjacent to 2. This proves our claim, by induction. Since 2 is also adjacent to 1, then 2 is of degree $n - 1$. But we know that G is a regular graph. Therefore, every vertex in G has degree $n - 1$. Consequently, G is the complete graph K_n.

Case 2. $g = 4 < n$. Let $[1, 2, 3, 4, 1]$ and $[1, 2, \ldots, n, 1]$ be 4- and n-cycles respectively in G. We claim that n is even and that $[j, j + 1]$, $[j, j + 3]$, $\ldots$, $[j, j + n - 1]$ are edges of G for each $j = 1, 2, \ldots, n$. Our claim can be easily verified in the case $4 \leq n \leq 7$. Let us then assume that $n \geq 8$. Consider the vertex $j = 1$. We shall prove by induction that the edges $[1, 2]$, $[1, 4]$, $[1, 6]$, $\ldots$ belong to G. Clearly, $[1, 2]$ is an edge. Assume that $[1, 2t]$ is an edge. By our Lemma, $[x, x + 3]$ is an edge for each vertex x. Hence, the path $[3, n, n-1, \ldots, 2t+3, 2t, 2t+1, 2t-2, 2t-1, \ldots, 4, 5, 2, 1]$ which has $n-1$ vertices belongs to G. It follows that 1 is adjacent to $2t+2$. We have therefore shown that 1 is adjacent to all the even numbered vertices. Consequently, n is even for otherwise we would get a cycle of length 3 in G. We have already shown that for $j = 1$, the edges $[j, j+1]$, $[j, j+3]$, . . . are all in G. Exactly the same argument can be used for $j = 2, 3, \ldots, n$.

Now, let A be the set of all vertices in G with odd labels and let B be the set of all vertices with even labels. Our result shows that each vertex in A is adjacent to each vertex in B. Furthermore, since the girth of G is 4, the vertices in A as well as the vertices in B are mutually non-adjacent. Therefore G is a complete bipartite graph. Since we know also that G must be regular, then A and B have the same cardinality, say m. Necessarily, $m \geqslant 2$ since G has cycles. Therefore, $n = 2m$ where $m \geqslant 2$ and G is the complete bipartite graph $K_{m,m}$.

References

[1] Gervacio, S. V. *On *-BD Graphs.* Research Report No. 29, 1976. Nanyang University, Singapore: Lee Kong Chian Institute of Math and Computer Science.

[2] ——————— *Block Design Graphs and Associated Hypergraphs.* Southeast Asian Bulletin of Mathematics, 1977.

[3] Koh, K. M. and Ho, Y. S. *On n-BD Graphs.* Research Report No. 13, 1976. Nanyang University Singapore: Lee Kong Chian Institute of Math and Computer Science.

[4] ——————— *A Characterization of n-BD Graphs Associated with Geometric Sequences. Research Report No. 16,* 1976. Nanyang University, Singapore: Lee Kong Chian Institute of Math and Computer Science.

[5] ——————— *On n-BD Graphs Representing the Sequence (k, m, 1, 1, . . ., 1).* Research Report No. 18, 1976. Nanyang University, Singapore: Lee Kong Chian Institute of Math and Computer Science.

Rolando E. Ramos, Discussant

In the paper entitled "n-cycle Block Design Graphs", Dr. Severino V. Gervacio introduced the concept of n-cycle BD graph. Then Dr. Gervacio showed five properties of n-cycle BD graphs, in particular, an n-cycle BD graph is hamiltonian. Finally, he characterized these graphs.

Firstly, what is one significance of Dr. Gervacio's results? These results have practical applications. For example, suppose a real estate developer wants to build a resort. For one reason or another, the resort should have four features, say, a golf course, a tennis court, a swimming pool and a massage clinic, and there should be exactly six ways of touring it. In other words, the developer wants to construct a 4-cycle BD graph with parameter $\lambda_1 = 6$. From Dr. Gervacio's results, the design of the resort should be similar to the complete graph K_4.

Lastly, what research problem can we formulate from Dr. Gervacio's paper? Let us define *n-path BD graphs* as follows: a connected graph G is called an n-path BD graph if a longest path in G is P_n and there exist positive integers λ_1, λ_2, $\ldots$, λ_n such that each path P_i ($i = 1, 2, \ldots, n$) in G is contained in exactly λ_i copies of P_n. Our problem is to characterize n-path BD graphs, that is, to find a necessary and sufficient condition for a graph to be an n-path BD graph. In solving this problem, we can follow the approach of Dr. Gervacio's paper.