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MINIMIZING THE MAXIMUM DIFFERENCE BETWEEN INTEGER LABELS OF ADJACENT VERTICES IN A GRAPH

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ABSTRACT

In 1962, F. Harary proposed the following problem. Given a graph G of order n, how can its vertices be labeled with the positive integers 1, 2, . . . , n such that the maximum difference between labels of adjacent vertices is a minimum?

We shall use the term *density* to denote the minimum of the maximum difference between adjacent labels in a graph of order n whose vertices are labeled with 1, 2, . . . , n. In this paper, we give formulas for densities of some classes of graphs as a function of their orders. In general, lower and upper bounds of the density of a graph in terms of graph parameters other than order are obtained. A note on the behavior of density with respect to the maximum degree of a graph is also included.

Introduction

In this study all graphs considered are finite, undirected, loopless and without multiple edges. If G is a graph, V(G) denotes its vertex-set and E(G) denotes its edge-set. The graph G is written as the ordered pair $G = \langle V(G), E(G) \rangle$.

The symbol $\lfloor x \rfloor$ denotes the greatest integer not exceeding x and the symbol $\lceil x \rceil$ denotes the smallest integer not less than x.

An *integer label* of a graph G is an assignment of distinct integers to the vertices of G. More formally, we define an integer label of G to be a one-to-one mapping $\lambda: V(G) \rightarrow Z$, where Z is the set of all integers.

Let λ be an integer label of a graph G, and let $a, b \in Z$ such that $a \neq 0$. We define the mapping $a\lambda + b$ by $(a\lambda + b)(v) = a\lambda(v) + b \quad v \in V(G)$. The following lemma is easy to prove.

LEMMA 1. Let λ be an integer label of a graph G and let $a, b \in Z$ such that $a \neq 0$. Then $a\lambda + b$ is also an integer label of G.

Let λ be an integer label of G. If $e = [a, b] \in E(G)$, we define the λ-*span* of e to be the positive integer $e_\lambda = |\lambda(a) - \lambda(b)|$. The *divergence* of λ in G is define to be $G_\lambda = \max\{ e_\lambda : e \in E(G) \}$. Finally, we define the *density* of G, denoted by p(G), to be the minimum G_λ, as λ ranges over all the integer labels of G. We assume that G has positive size (number of edges) in the definitions of divergence and density. For convenience, we define the density of a graph having size zero to be zero.

LEMMA 2. Let G be a graph. Then there exists an integer label λ of G such that λIV(G)) consists of consecutive integers and $G_\lambda = \rho(G)$.

Proof: Let ϕ be an integer label of G such that $G_\phi = \rho(G)$. If $v_1, v_2, \ldots$, vn are the vertices of G, we may assume, without loss of generality, that $\phi(v_1) < \phi(v_2) < \ldots < \phi(v_n)$. If $\phi(V(G))$ does not consist of consecutive integers, we shall show how to construct one which does. Assume that $\phi(v_k)$ and $\phi(v_k)$ and $\phi(v_{k+1})$ are not consecutive integers, and let the positive difference between them be t. Define the integer label ϕ' by $\phi'(v_i) = \phi(v_i)$ if $i \leq k$ and $\phi'(v_i) = \phi(v_i) - t + 1$ otherwise. Obviously, the ϕ'-span of any edge is less than or equal to its ϕ-span. Hence, the divergence of ϕi is less than or equal to that of ϕ. But ϕ has the minimum divergence. Therefore, ϕ' has the same divergence as ϕ. Note that $\phi'(v_k)$ and $\phi'(v_{k+!)}$ are consecutive integers already. This procedure may be repeated a sufficient number of times until we obtain the desired integer label λ.

Using Lemma 1, we easily deduce the following corollary.

COROLLARY. If G is a graph of order n, then there exists an integer label λ of G such that $G_\lambda = \rho(G)$ and $\lambda(V(G))$ consists of $1, 2, \ldots, n$.

We define a ***natural integer label*** of a graph G of order n to be an integer label λ such that $G_\lambda = p(G)$ and $\lambda(V(G))$ consists of the integers $1, 2, \ldots, n$.

This study deals mainly with the problem of finding natural integer labels of graphs – a problem proposed by Harary in 1962.

LEMMA 3. Let G be a graph and H a subgraph of G. Then $\rho(H) \leq p(G)$.

Proof: Let λ be a natural integer label of G. Then the restriction $\lambda|_H$ of λ to H is an integer label of H whose divergence is less than or equal to $\rho(G)$. It follows $p(H) \leq p(G)$.

LEMMA 4. Let G be a graph whose connected components are $G_1, G_2, \ldots, G_k$. Then $g(G) = \max\{\rho(G_i)\}$.

Proof: Observe that if λ is an integer label of a graph H, then the mapping λ +b, where b is any integer, is also an integer label of H with the same divergence as λ. Furthermore, if $\lambda(V(H))$ consists of consecutive integers, then $(\lambda + b)(V(H))$ also consists of consecutive integers. Let λ_1 be a natural integer label of G_1. In view of our observation, we can find an integer label λ_2 of G_2 whose divergence is $\rho(G_2)$ and such that $\lambda 2(V(G_2))$ consists of consecutive integers the smallest of which is equal to the order of G_1 plus 1. We see therefore that integer labels λ_i can be found such that the divergence of λ_i is $\rho(G_i)$ and that the union of all $\lambda_i(V(G_i))$ is $\{1, 2, \ldots, n\}$, where n is the order of G. The mapping λ whose restriction to G_i, for each i, is λ_i is then an integer label of G whose divergence is equal to $\max\{\rho(G_i)\}$. Therefore $\rho(G) \leq \max\{\rho(G_i)\}$. Now, since each G_i is a subgraph of G, $\rho(G_i) \leq \rho(G)$. It follows that $\max\{\rho(G_i)\} \leq \rho(G)$. Therefore, $\rho(G) = \max\{\rho(G_i)\}$.

In view of the preceding lemma, we need to consider only the problem of finding natural integer labels of connected graphs.

The formulas in the next lemma are easy to derive. Part (d) may be established with the help of the idea presented in the proof of Theorem 3.

LEMMA 5.

(a) $\rho(P_n)$ = 1 for all paths P_n with $n \geq 2$.

(b) $\rho(C_n)$ = 2 for all cycles C_n with $n \geq 3$.

(c) $\rho(K_n)$ = $n-1$ for all complete graphs K_n with $n \geq 2$.

(d) $\rho(K_{m,n}$ = $[(m+1)/2] + n - 1$ for all complete bipartite graphs $K_{m,n}$ with $m \geq n$.

COROLLARY. If G is a graph with maximum clique of order q, then $\rho(G) \geq q-1$.

Proof: This follows from Lemma 3 and Lemma 5 (c).

Note that this lower bound is attained by all paths of order greater than 2, and hence is a best possible lower bound.

Bounds for density in terms of maximum degree or minimum degree

THEOREM 1. Let G be a graph with maximum degree Δ. Then $\rho(G) \geq [(\Delta+1)/2]$.

Proof: Since G has maximum degree Δ then G contains a subgraph isomorphic to $K_{\Delta, 1}$ whose density, by Lemma 5, is $[(\Delta+1/2]$. By Lemma 3 (d), $\rho(G)$, $\geq [(\Delta+1)/2)$.

By Lemma 5 (c), the density of $K_{1,n}$ is $[(n+1)/2]$. Since n is the maximum degree of this graph, we see that the lower bound given in Theorem 1 is best possible.

LEMMA 6. Let G be a graph of order n, v a vertex of G, and $N_v = \{\lambda: \lambda$ is a natural integer label of G with $\lambda(v)= n\}$. Then $G_\lambda \geq \deg(v)$ for each $\lambda \in N_v$.

Proof: Let $\deg(v) = d$ and let $v_1, v_2, \ldots, v_d$ be all the neighbors of v. Let $\lambda \in N_v$ and without loss of generality we assume that $\lambda(v_1) < \lambda(v_2) < \ldots < \lambda(v_d)$. Since $\lambda(v) = n$, then $\lambda(v_d) \leq n-1$ and $\lambda(v_1) \leq n-d$. Therefore the λ-span of the edge $[v, v_1]$ is at least $n-(n-d) = d$. Therefore $G_\lambda \geq d = \deg(v)$.

THEOREM 2. Let G be a graph with minimum degree σ. Then $\rho(G) \geq \sigma$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $N_i = \{\lambda: \lambda$ is a natural integer label of G such that $\lambda(v_i) = n\}$. Then $N = \cup N_i$ is the set of all natural integer labels of G. Now, $\rho(G) = \min\{G_\lambda: \lambda \in N\} = \min\{\min\{G_\lambda: \lambda \in N_i\}: i = 1, 2, \ldots, n\} \geq \sigma$.

The *complement* of G, denoted by $\bar{G}$, is the graph with vertex set V(G) and where two distinct vertices are adjacent if and only if they are not adjacent in G.

COROLLARY 1. Let G be a graph with maximum degree Δ and minimum degree σ. Then $\rho(G) + \rho(\tilde{G}) \geq n-1-(\Delta-\sigma-)$

Proof: The minimum degree of $\tilde{G}$ is $n-1-\Delta$. Thus, applying Theorem 2, we get the desired result.

If G is an r-regular graph, then $\sigma = \Delta = r$ and we get the following corollary.

COROLLARY 2. If G is a regular graph, then $\rho(G)+(\rho(\tilde{G}) \geq n-1$.

The lower bound for density given in Theorem 2 is also best possible since it is attained by the cycle C_n. In fact, if $n > 3$, the complement of the cycle C_n also attains the same lower bound, namely $\sigma = n-3$.

Bound for density in terms of stability number

THEOREM 3. Let G be a graph of order n and stability number $\alpha(G)$. Then $\rho(G) \leq n - 1 - [\alpha(G)/2]$.

Proof: Let S be a maximum stable set in G, i.e., $|S| = \alpha(G)$. Let λ be any integer label of G such that $\lambda(V(G)) = \{1, 2, \ldots, n\}$ and $\lambda(S)$ consists of the $[\alpha(G)/2]$ smallest labels and the $[a(G)]$ largest labels, i.e., $\lambda(G) = \{1, 2, \ldots, [\alpha(G)/2], n, n-1, \ldots, n-[\alpha(G)/2]+1\}$. Then no edge can have a λ-span of $n-[\alpha(G)/2]$ or more. Hence, $\rho(G) \leq n-1-[\alpha(G)/2]$.

The upper bound for density given in Theorem 3 is best possible since it is attained by the graph $K_{1,n}$.

THEOREM 4. Let G be a graph with stability number α and minimum degree σ. Then $\rho(G) + \rho(\widetilde{G}) \geq \alpha + \sigma - 1$.

Proof: Clearly, the complement of G contains a clique of order α. Hence, combining Theorem 2 and the Corollary to Lemma 5, we get the desired result.

Bound for density in terms of diameter

LEMMA 7. Let $P = x_1 x_2 \ldots x_k x_{k+1}$ be a path of length k and let λ be an integer label of P such that $\lambda(x_1) = 1$, $\lambda(x_{k+1}) = n$ and $1 \leq \lambda(x_i) \leq n$ for each i. Then the divergence P_λ is at least $[(n-1)/k]$.

Proof: For convenience, we let $\lambda(x_i) = a_i$. In case $\langle a_i \rangle$ is an increasing sequence, then the summation of all λ-spans is n-1. Otherwise, we consider the maximal monotonic subsequences of $\langle a_i \rangle$:

$$
\begin{array}{l}
a_1 < a_2 < \ldots < a_{n1} \\
a_{n1} > a_{n1+1} > \ldots > a_{n2} \\
a_{n2} < a_{n2+1} < \ldots < a_{n3} \\
\quad ; \\
\quad ; \\
a_{np} < a_{np+1} < \ldots < a_n
\end{array}
$$

In this case, we see that the sum of all λ-spans is $(a_{n1} - a_1) + (a_{n1} - a_{n2}) + (a_{n3} - a_{n2}) + \ldots + (a_n - a_{np})$. Clearly, this quantity is greater than $n - 1$. Thus, in all cases, the average λ-span is at least $[(n-1)/k]$. It follows that the maximum λ-span is at least $[(n-1)/k]$. Consequently, the divergence is at least $[(n-1)/k]$.

THEOREM 5. Let G be a connected graph of order n and diameter d. Then $\rho(G) \geq [(n-1)/d]$.

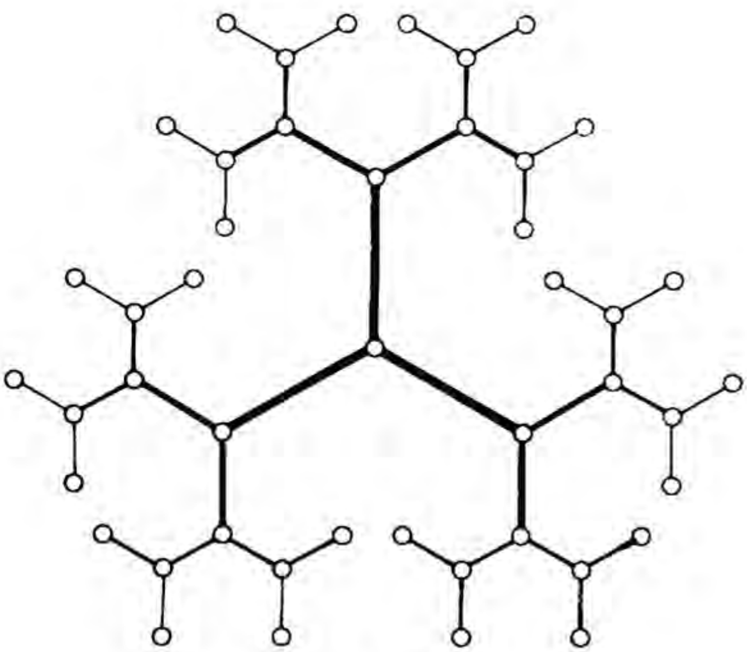
Proof: Let λ be any natural integer label of G. Let u, v be the vertices with $\lambda(u) = 1$ and $\lambda(v) = n$. Let P be any shortest path joining u and v. Then by Lemma 7, there is an edge of P whose λ-span is at least $[(n-1)/k]$, where k is the length of P. But $k \leq d$ since d is the maximum length of a path in G. Hence $[(n-1)/k] \geq [(n-1)/d]$. It follows that $\rho(G) \geq [(n-1)/d]$.

Observe that the lower bound for density in Theorem 5 is attained by the path $P_n (n>1)$, the cycle C_n $(n>2)$, and the complete graph $K_n (n>1)$.

Further result

This section aims to show that it is impossible to get an upper bound for density in terms of maximum degree only. Specifically, given a fixed $\Delta > 2$, a graph with maximum degree Δ may have an arbitrarily large density. We will illustrate this for $\Delta = 3$ by showing that given any positive integer N, there exists a graph G of maximum degree 3 whose density is greater than N.

We define $G_1 = K_{1,3}$. Recursively, we define G_k as follows. Take as many mutually disjoint copies of $K_{1,2}$ as there are end vertices of G_{k-1} $(k>1)$. To each end vertex of G_{k-1}, attach the central vertex of one $K_{1,2}$. The resulting graph is G_k. It is easy to see that G_k has diameter 2k and simple mathematical induction will show that it has order $3 \cdot 2^k - 2$. As an example, the graph G_4 is shown below.



GRAPH G4

By Theorem 5, $\rho(G_k) \geq [(n-1)/2k] = [3(2^k-1)/2k]$. Hence, given any integer $N > 0$, we choose k to be any integer which is greater than $2 \cdot \ln(N)/\ln(2)$. Then $\rho(G_k) \geq [3(s^k-1)/2k] > N$.

Reference

[1] Theory of Graphs and its applications. *Smolenice Symposium* (Prague, 1964).