Which Bipartite Graphs Are Singular?

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ABSTRACT

The subdivision graph s(G) of any graph G is a bipartite graph and this paper gives a necessary and sufficient condition for the subdivision graph of a connected graph to be singular.

INTRODUCTION

By a *graph* we shall understand a pair G = <V(G), E(G)> where V(g) is a finite nonempty set and where E(G) is a set of unordered pairs of distinct elements of V(G). The elements of V(G) are called *vertices* while those of E(G) are called *edges*. We say that the vertices x and y are *adjacent* if the unordered pair (edge) {x, y} ε E(G). Other terms and concepts will be defined when the need arises. The reader is referred to [1], [2] and [3] for other definitions not given here.

If G is a graph of order n with vertices 1, 2, . . ., n then its *adjacency matrix*, denoted by A(G), is the nxn matrix (a_{ij}) where aij = 1 if i and j are adjacent ({1,j} ε E(G)) and a_{ij} = 0 otherwise. The graph G is said to be *singular* if its adjacency matrix is singular, i.e., det A(G) = 0. Otherwise, we call G a *nonsingular* graph.

The study of singular graphs is of particular importance in organic chemistry [5]. Conjugated hydrocarbons are associated with graphs whose vertices correspond to the carbon atoms and whose edges correspond to chemical bonds between these atoms. Other relationships between chemistry and graphs are outlines in [5].

Here we shall give some basic results pertaining to singularity of graphs and apply them to prove results on the singularity or nonsingularity of bipartite graphs.

PRELIMINARY RESULTS

A subset S of the vertex-set V(G) of a graph G is said to be *stable* if no two vertices in S are adjacent. The maximum number of vertices in graph which form a stable set is called the *stability number* of the graph, usually denoted by α. Thus, the path P_n has stability number $\alpha = \left[\frac{n}{2}\right]$; the *cycle* C_n has $\alpha = \left[\frac{n}{2}\right]$; the *complete graph* K_n has $\alpha = 1$; the *complete bipartite graph* $K_{m,n}$ has $\alpha = \max\{m,n\}$ and the totally disconnected graph $\overline{K}_{m,n}$ has $\alpha = n$.

Example 1. Fig. 1 shows some graphs with their corresponding stability numbers.

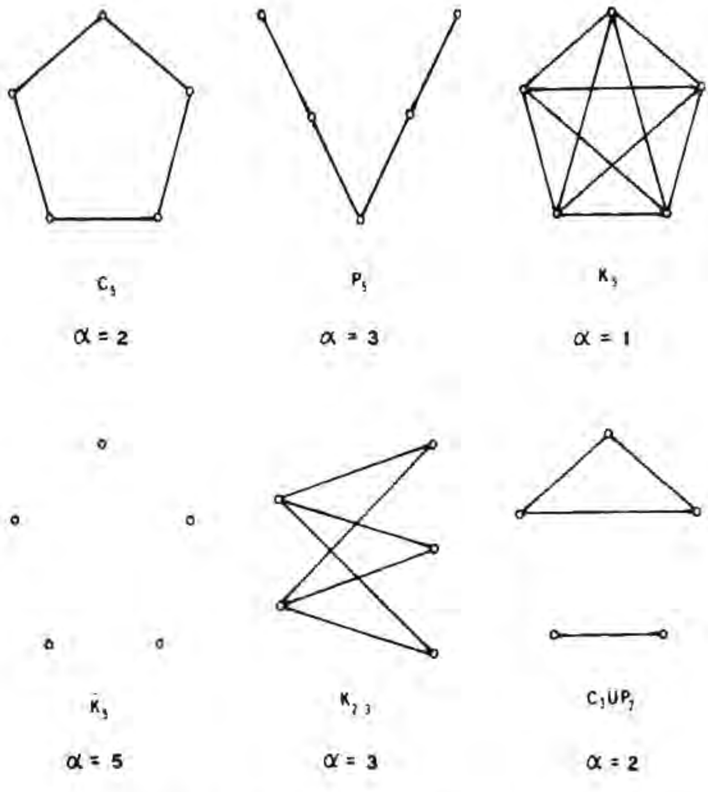


Fig. 1 The stability numbers of some graphs.

Remark. Any graph is either *connected* or *disconnected.* We shall not anymore give the precise definition of connected graph. However a very good idea of the concept may be obtained by knowing that all the graphs in Fig. 1 are connected except for the last two.

Theorem 1. *Let G be a graph of order n and stability number a. If* $a > \frac{n}{2}$ *then G is singular.*

The proof of Theorem 1 can be found in [4].

Theorem 2. *If there exist two distinct vertices in a graph G with identical neighbor-sets, then G is singular.*

Proof: Let u and v be distinct vertices in the graph G having identical neighbor-sets. Then, it is quite obvious that the rows corresponding to u and v in A(G) are identical. Therefore, det A(G) = 0 and G is singular.

Many classes of graphs have been investigated for singularity. Among those reported in [5] are the following results:

(1) The path P_n is singular if and only if n is odd.
(2) The even cycle C_{2k} is singular if and only if k is even.
(3) The complete bipartite graph $K_{m,n}$ is singular if and only if $m + n > 2$.
(4) All bipartite graphs of odd order are singular.

A result not included in [5] which is easy to prove is that the odd cycle C_{2k+1} is nonsingular and det $A(C_{2k+1}) = 2$. Also, if k is odd, then det $A(C_{2k}) = -4$.

SOME RESULTS ON BIPARTITE GRAPHS

A graph G is said to be *bipartite* if V(G) can be partitioned into two nonempty stabile sets. The nontrivial trees (connected graph of order at least 2 and having no cycles) form an important subset of the set of all bipartite graphs. In [4], nonsingular trees are completely characterized. From this, we get the following characterization of singular trees.

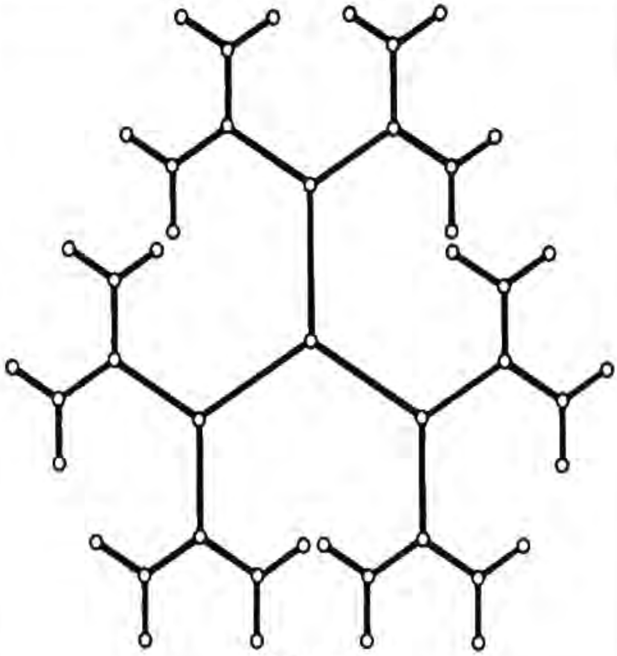
Theorem 3. *Let T be a tree of order n.*

(i) If n is odd then T is singular.

(ii) If n is even then T is singular if and only if there exists a vertex v in T such that the graph T-v has at least two components of odd orders.

Remark. It can be shown by mathematical induction that if T is a nonsingular tree of even order n = 2k then det $A(T) = (-1)^k$.

An *end-vertex* of a graph G is a vertex of degree 1, *i.e.*, it is a vertex with exactly one neighbor.



The Graph G_4

Lemma 1. *Let x be an end-vertex of a graph G whose unique neighbor, say y, has degree 2. Let G* be the graph obtained from G by deleting x and y. Then det(G*) = - det A(G).*

Proof: Let 1, 2, . . . , n denote the vertices of G and without loss of generality, let x = 1, y = 2. Let the other neighbor of y be the vertex 3. Then the adjacency matrix of G has the following form:

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 1 & \cdots & & & & & \\ 0 & 0 & \vdots & & & & & \\ . & . & \vdots & & A(G^*) & & & \\ . & . & \vdots & & & & & \\ 0 & 0 & \vdots & & & & & \end{bmatrix}$$

If we substract row 1 from row 3 and then interchange rows 1 and 2, we see that the resulting matrix has determinant equal to det A(G*). But because of the row interchange, we have reversed the sign of the value of the original determinant. It follows that det $A(G^*)$ = -det $A(G)$.

Let us now consider one easy operation which creates a bipartite graph out of any graph. If [x, y] is an edge of a graph, we insert a new vertex z in this edge by replacing the edge [x, y] by the two edges [x, z] and [z, y]. This process, called *subdivision* of the edge [x, y], is illustrated in Fig. 2.

Fig. 2. Subdividing the edge (x, y).

Let G be a graph of order n and size m (having m edges). The *subdivision graph* of G, denoted by s(G), is the graph obtained from G by subdividing each edge of G. Thus, $s(P_n) = P_{2n-1}$ and $s(C_n) = C_{2n}$. It is quite clear that for any graph G, s(G) does not contain cycles of odd length. Consequently, s(G) is always a bipartite graph provided that G is nontrivial.

It is easy to see that a graph G is singular if and only if at least one of its component is singular. In view of these, we shall now consider only connected graphs. The main result we shall give here is a necessary and sufficient condition for the subdivision graph s(G) of a connected graph G to be singular. We shall prove three separate theorems which we can combine later to obtain our main result.

Theorem 4. *Let G be a connected graph. If G has no cycles (G is a tree) then s(G) is singular.*

Proof. The size of G is n-1 and so the subdivision graph s(G) is of order 2n-1. Clearly, the original n vertices of G form a stable set in s(G). Therefore, the stability number α of s(G) satisfies $\alpha \geq n > \frac{2n-1}{2}$. By Theorem 1, G is singular.

Theorem 5. *Let G be a connected graph with exactly one cycle (G is unicyclic). Then s(G) is singular if and only if G is bipartite.*

Proof. Let C_p be the unique cycle of G. Then s(G) has a unique cycle, namely C_{2p}. By a repeated application of Lemma 1 to s(G), we see that s(G) may be reduced to C_{2p}, and so det $A(s(G)) = \pm$ det $A(C_{2p})$. But we know that C_{2p} is singular if and only if p is even. Therefore, s(G) is singular if and only if G is bipartite.

Theorem 6. *Let G be a connected graph with at least two cycles. Then s(G) is singular.*

Proof. Let G be of order n and size m. Since G has at least two cycles, then m > n. Clearly, the m new vertices in s(G) form a stable set in s(G). Therefore, the stability number α of s(G) satisfies $\alpha \geq$ m. But s(G) has order m+n and m > m+n since m > n. Therefore, by Theorem 1, s(G) is singular.

Observe that the last three theorems exhaust all the possible cases for a connected graph in terms of the number of cycles. Observe further that in most cases, s(G) is singular. It is therefore convenient to summarize the three theorems by characterizing nonsingular subdivision graphs.

Theorem 7. *Let G be a connected graph. Then s(G) is nonsingular if and only if G is unicyclic and nonbipartite.*

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