Singularity of Graphs in Some Special Classes[1]

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ABSTRACT

*A **graph** is a pair $G = <V(G), E(G)>$, where $V(G)$ is a nonempty finite set of elements called **vertices** and $E(G)$ is a set of unordered pairs of distinct vertices called **edges**. If $v_1, v_2, \ldots, v_n$ are the vertices of G, we define the **adjacency matrix** of G, denoted by $A(G)$, to be the nxn (0, 1) - matrix (a_{ij}), where $a_{ij} = 1$ if and only if $[v_i, v_j] \in E(G)$. The graph G is said to be **singular** if its adjacency matrix is singular, i.e., det $A(G) = 0$.*

*Singular graphs have not yet been characterized and the identification of all singular graphs seems to be a difficult problem. However, characterization of singular graphs in some special classes is possible. Here we shall completely characterize the singular graphs among the **planar grids** $P_m \times P_n$, the **prisms** $C_m \times P_n$ and the **toroidal grids** $C_m \times C_n$.*

Introduction

The *path of order n*, denoted by P_n, is the graph with n vertices 1, 2, . . . , n and whose edges are [i, i + 1], i = 1, 2, 3,..., n - 1. The *cycle of order n*, denoted by C_n, is the graph obtained from P_n by adding the edge [1, n]. Figure 1 shows the path P_6 and the cycle C_6.

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[1] The results contained in this paper are taken from the NRCP-funded research project entitled "A Study of Singular Bipartite Graphs."

If $G = \langle V(G), E(G) \rangle$ and $H = \langle V(H), E(H) \rangle$ are two graphs, the *cartesian product* **G** x **H** is the graph with vertex-set **V(G)** x **V(H)**, and two vertices (a, b) and (c, d) in **G** x **H** are adjacent if and only if either (i) [a, b] ε **E(G)** or (ii) a = c and [b, d] ε **E(H)**. Figures 2, 3 and 4 show the planar grid $\mathbf{P_5} \times \mathbf{P_8}$, the prism $\mathbf{C_6} \times \mathbf{P_4}$ and the toroidal grid $\mathbf{C_4} \times \mathbf{C_6}$, respectively.

In this paper, we shall determine which planar grids, prisms and toroidal grids are singular. Some reduction formulas [1] are available to handle the planar grids. However, we shall use a uniform procedure in handling all the three classes. We shall first establish one Lemma which will help us do this. The following notations are used in the statement and proof of the Lemma:

P(a, b)	denotes the point P in the plane with coordinates (a, b).
PQ	is the line segment with endpoints P and Q.
I PQI	is the length of the line segment PQ.
gcd (a, b)	is the greatest common divisor of a and b.

PRELIMINARY RESULT

Lemma 1. Let P(a, b) and Q(c, d) be any two distinct points in the plane with integer coordinates. Then the number of points in PQ with integer coordinates (including P and Q) is equal to 1 + gcd(c-a, d-b). Furthermore, these points are evenly distributed over the line segment PQ, i.e., the distance between any two such neighboring points is I PQ I /gcd(c-a, d-b).

Proof: If the line segment PQ is horizontal or vertical, the Lemma clearly holds. We, therefore, assume that PQ is neither horizontal nor vertical. Without loss of generality, assume that $c > a$ and $d > b$ and let $g = \gcd(c-a, d-b)$. Let $0 \leq k \leq g$ and $x = a + k(c-a)/g$, $y = b + k(d-b)/g$. It is easy to check that R(x, y) is a point in PQ with integer coordinates and that the distance between two such neighboring points is I PQI /g. Since these points are g + 1 in number, it remains for us to show that there are no other points in PQ with integer coordinates. To prove this, let S(u, v) be any point in PQ with integer coordinates. Please refer to Figure 5.

Without loss of generality, assume that S is not the point P. Since g = gcd(c-a, d-b), then gcd(m, n) = 1, where m = (c-a)/g and m = (d-b)/g. By similar triangles, we have (v-b)/(u-a) = (d-b)/(c-a) = m/n. It follows that u-a = kn and v-b = km for some $0 \leq k \leq g$. Consequently, S is one of the points R(x, y).

In addition to the above Lemma, we shall use the following results on eigenvalues which can be found in [ ]:

(a) The eigenvalues of $\mathbf{A(P_m \times P_n)}$ are
$2\cos[\pi/(m+1)]i + 2\cos[\pi/(n+1)]j \quad (1 \leq i \leq m \text{ and } 1 \leq j < n)$.

(b) The eigenvalues of $\mathbf{A(C_m \times P_n)}$ are
$2\cos(2\pi/m)i + 2\cos[\pi/(n+1)]j \quad (1 \leq i \leq m \text{ and } 1 \leq j < n)$.

(c) The eigenvalues of $\mathbf{A(C_m \times C_n)}$ are
$2\cos(2\pi/m)i + 2\cos(2\pi/n)j \qquad (1 \leq i \leq m \text{ and } 1 \leq j < n)$.

SINGULAR PLANAR GRIDS

Using (a), we see that $\mathbf{P_m \times P_n}$ is singular if and only if $2\cos[\pi/(m+1)]i + 2\cos[\pi/(n+1)]j = 0$ for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. Using trigonometric identity $\cos\alpha + \cos\beta = 2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$, we see that the planar grid is singular if and only if $\cos 1/2[(\pi/m+1)i + (\pi/n+1)j] = 0$ for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. But $1/2[(\pi/m+1) - (\pi/n+1)j]$ lies in the interval $(-\pi/2, \pi/2)$ and cosine is never zero here. On the other hand, $1/2[(\pi/m+1)i + (\pi/n+1)j]$ is in the interval $(0, \pi)$ and cosine is zero only at the point $\pi/2$. Hence, 0 is an eigenvalue of $\mathbf{A(P_m \times P_n)}$ if and only if $1/2[(\pi/m+1)i + (\pi/n+1)j] = \pi/2$ for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. This necessary and sufficient condition easily reduces to the following:
$[(i/m+1) + (j/n+1)] = 1$ for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. Observe that the equation $[(i/m+1) + (j/n+1)] = 1$ represents a straight line in the ij-plane with i - and j - intercepts of m + 1 and n + 1, respectively. We see then that there exists i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$ if and only if there is at least one point in the line segment joining the i - and j - intercepts with integer coordinates. By Lemma 1, there is at least one such point if and only if $\gcd(m+1, n+1) > 1$. We have thus established the following:

Theorem 1. The planar grid $\mathbf{P_m} \times \mathbf{P_n}$ is singular if and only if $\gcd(m+1, n+1) > 1$.

SINGULAR PRISMS

Using (b) and the same trigonometric identity applied in the proof of Theorem 1, we see that $\mathbf{C_m} \times \mathbf{P_n}$ is singular if and only if $\cos 1/2[(2\pi/m)i \pm (\pi/n+1)] = 0$ for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. Now, $1/2[(2\pi/m)i + (\pi/n+1)]$ is in the interval $[0, (3/2)\pi)$ while $1/2[(2\pi/m)i - (\pi/n+1)]$ is in the interval $(-\pi/2, \pi)$. In both intervals, cosine is 0 only at the point $\pi/2$. Hence, $\mathbf{C_m} \times \mathbf{P_n}$ is singular if and only if (i) $[(2/m)i + (i/n+1)j] = 1$ or (ii) $[(2/m)i - (1/n+1)j] = 1$ for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. The graph of (i) in the ij-plane is a straight line passing through the points $P(0, n+1)$ and $Q(m, -(n+1))$. Since PQ cuts the i-axis at $(m/2, 0)$, it follows that (i) holds for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$ if and only if there are at least four points in PQ with integer coordinates. By Lemma 1, this is equivalent to the condition $\gcd(m, 2n+2) > 2$. Similarly, the graph of (ii) in the ij-plane is a straight line containing the points $P(0, -(n+1))$ and $Q(m, n+1)$. PQ also cuts the i-axis at $(m/2, 0)$ and so (ii) holds for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$ if and only if there are at least four points in PQ with integer coordinates. This condition also yields the equivalent to the condition $\gcd(m, 2n+2) > 2$. Therefore, we have established the following:

Theorem 2. The prism $\mathbf{C_m} \times \mathbf{P_n}$ is singular if and only if $\gcd(m, 2n+2) > 2$.

Remark. Theorem 2 is equivalent to the following:

Theorem 2'. The prism $\mathbf{C_m} \times \mathbf{P_n}$ is singular if and only if $m \equiv 0 \pmod 4$ and n is odd.

SINGULAR TOROIDAL GRIDS

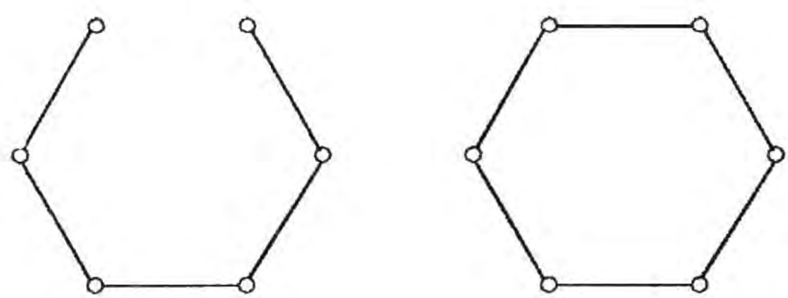
By means of (c) and the trigonometric identity used in Theorems 1 and 2, we obtain the result that $\mathbf{C_m} \times \mathbf{C_n}$ is singular if and only if $\cos[(\pi/m)i \pm (\pi/n)j] = 0$ for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$. But $[(\pi/m)i + (\pi/n)j]$ is in the interval $(o, 2\pi)$ while $[(\pi/m)i - (\pi/m)j]$ is in the interval $(-\pi, \pi)$. In the first interval, cosine is 0 at $\pi/2$ and $3\pi/2$ while in the second interval, cosine is 0 at $-\pi/2$ and $\pi/2$. From these, we see that $\mathbf{C_m \times C_n}$ is singular if and only if for some i and j satisfying $1 \leq i \leq m$ and $1 \leq j \leq n$, either one of the following conditions hold:

$$\text{(i)} \quad \frac{2ni + 2mj}{mn} = 1 \text{ or } 3$$

$$\text{(ii)} \quad \frac{2ni - 2mj}{mn} = 1 \text{ or } -1.$$

The numerator of (i) is always even while its righthand side is odd. Hence, (i) has no solution if m and n are both odd. The same conclusion holds for (ii). If one of m, n is even, we may assume without loss of generality that m is even. Taking $i = m/2$ and $j = n$, we will satisfy (i). Hence, we have proved the following:

Theorem 3. The toroidal grid $\mathbf{C_m \times C_n}$ is singular if and only if m or n is even.



P_6 C_6

Figure 1. The path P_6 and the cycle C_6

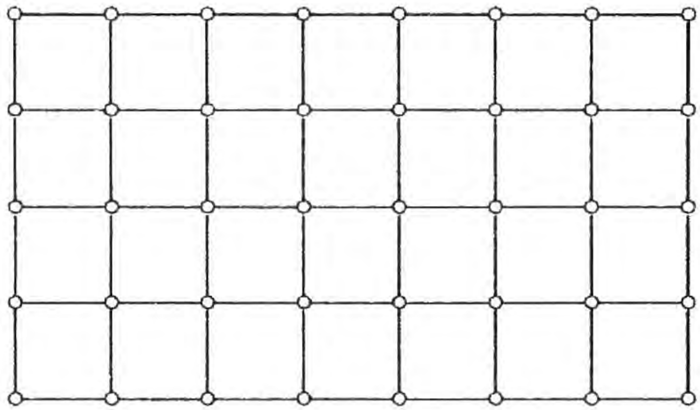


Figure 2. The planar grid $P_5 \times P_8$

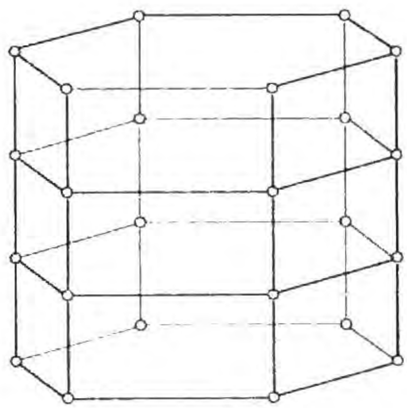


Figure 3. The prism $C_6 \times P_4$

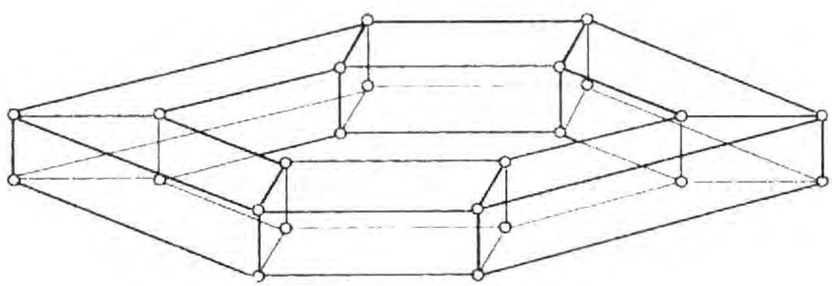


Figure 4. The toroidal grid $C_4 \times C_6$

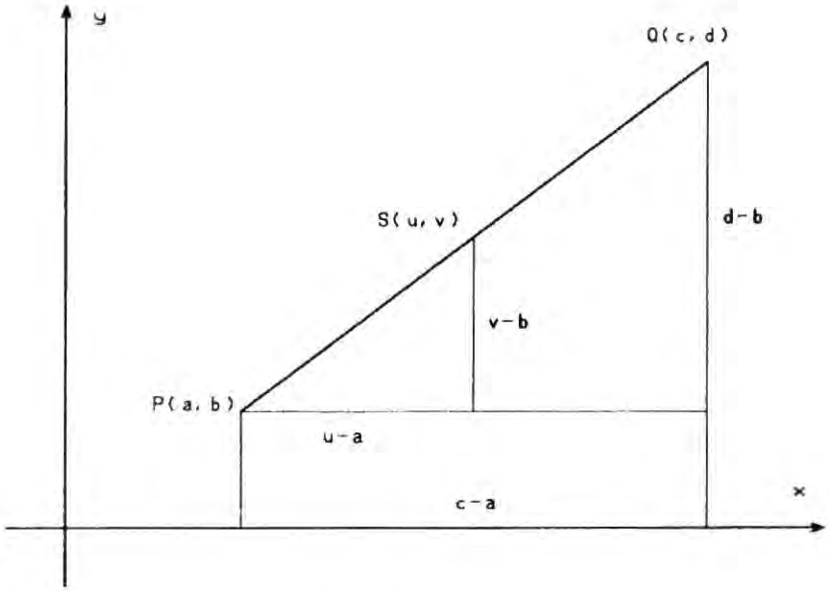


Figure 5. Three collinear points P, Q and S with integer coordinates

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