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SINGULAR GRAPHS: THE CARTESIAN PRODUCT OF TWO GRAPHS

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Abstract

The *adjacency matrix* of a graph G with vertices $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The graph G is said to be *singular* if $A(G)$ is singular, *i.e.*, det $A(G) = 0$; otherwise, G is said to be *non-singular*.

The Cartesian product of two graphs G and H, denoted by $G \times H$, may be singular or non-singular, independently of the singularity or non-singularity of G and H

The graph with n vertices where each vertex is adjacent to the remaining $n-1$ vertices is called the complete graph of order n, denoted by K_n. It is known that det $A(K_n) = (-1)^{n-1}(n-1)$ and hence Kn is non-singular only when $n \geq 2$. If G is any graph, we prove that $G \times K_n$ is singular if and only if 1 or $1-n$ is an eigenvalue of $A(G)$. In particular, we show that the Cartesian product of the cycle C_m and the complete graph K_n, $n \geq 4$, is singular if and only if $m = 0$ (mod 6). We also prove that det $A(K_m \times K_n) = (-2)^{(m-1)(n-1)}(m-2)^{n-1}(n-2)^{m-1}(m+n-2)$. As a corollary, $K_m \times K_n$ is singular if and only if m=2 or n=2 or m=n=1.

Introduction

By a graph G we shall understand a pair $<V(G), E(G)>$, were $V(G)$ is a finite, non-empty set of elements called *vertices* and $E(G)$ is a set of 2-subsets of $V(G)$ whose elements are called *edges*. The numbers $|V(G)|$ and $|E(G)|$ are called the *order* and *size* of G, respectively. For simplicity, an edge $\{x, y\}$ will be written as xy and we shall say that x and y are *adjacent*.

If G is a graph of order n having vertices $x_1, x_2, \ldots, x_n$, we define the *adjacent matrix* of G to be the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if x_i and x_j are adjacent; $a_{ij} = 0$, otherwise. The graph G is said to be *singular* if $A(G)$ is singular, *i.e.*, det $A(G) = 0$; otherwise, G is said to be *non-singular*.

The path of order n, denoted P_n, is the graph with vertices $x_1, x_2, \ldots, x_n$ and edges x_ix_{i+1}, i=1, 2,..., $n-1$. It is easy to prove that P_n is singular [3, 8] if and only if n is odd. The *cycle* of order $n \geq 3$, is the graph obtained from the path P_n by adding the edge $x_n x_1$. It is known [3, 8] that C_n is singular if and only if $n \equiv 0 \pmod 4$. The *complete graph* of order n, denoted by K_n, is the graph with n vertices such that every pair of distinct vertices forms an edge. There are several ways [3, 4, 6, 7] of showing that det $A(K_n) = (-1)^{n-1}(n-1)$ and hence, K_n is non-singular if and only if $n \geq 2$.

The *Cartesian product* of two graphs G and H, denoted by $G \times H$, is the graph with $V(G \times H) = V(G) \times V(H)$ and where two vertices (a,b) and (c,d) are adjacent if and only if (1) $a = c$ and $bd \in E(H)$ or (2) $b = d$ and $ac \in E(G)$. The Cartesian product $G \times H$ is easily constructed as follows: Consider the graph H. Replace each vertex of H by a copy G. If G_1 and G_2 are copies of G corresponding to adjacent vertices in H, we join by an edge each vertex of G_1 to the corresponding vertex in G_2. The graph constructed in this manner is $G \times H$. In this contruction, the roles of G and H may be interchanged since $G \times H$ is isomorphic to $H \times G$.

Three of the most frequently studied graphs are P_n, C_n, and K_n. Conditions which are both necessary and sufficient for their singularity are known. How about the Cartesian product of two (not necessarily in different classes) of these graphs? The characterization of singular graphs in each of the classes $P_m \times P_n$, $C_m \times P_n$ and $C_m \times C_n$ has been completely solved [5]. Agpalza and Gervacio [1] proved that $P_m \times K_n$ is singular if and only if $m \equiv 0 \pmod 4$. We study next the graphs $C_m \times K_n$. $C_m \times K_1$ is C_m and singular C_m's have been characterized [3, 7, 8]. $C_m \times K_2$ is $C_m \times P_2$ and this belongs to the class $C_m \times P_n$ of which the singular graphs have been identified [5]. $C_m \times K_3 = C_m \times C_3$ and this is covered in $C_m \times C_n$ [5]. Thus, insofar as $C_m \times K_n$ is concerned, the only case that remains is $n \geq 4$. Agpalza [1] proved $C_m \times K_4$ is singular if and only if $m \equiv 0 \pmod 6$.

Main Results

Lemma 1. *Let P and Q be r x r matrices and let M be an n x n block matrix with diagonal elements all equal to P and all other elements equal to Q. Then* det M = det $(P + (n-1)Q)(\det(P-Q))^{n-1}$.

Proof: We perform elementary operations on the (block) rows and (block) columns of M as follows: Add each of the last $n-1$ rows of M to row 1. This yields a matrix M' where each entry in row 1 is equal to $R = P + (n-1)Q$ as shown in Figure 1.

$$\begin{pmatrix} P & Q & Q & \dots & Q \\ Q & P & Q & \dots & Q \\ Q & Q & P & \dots & Q \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ Q & Q & Q & \dots & P \end{pmatrix} \longrightarrow \begin{pmatrix} R & R & R & \dots & R \\ Q & P & Q & \dots & Q \\ Q & Q & P & \dots & Q \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ Q & Q & Q & \dots & P \end{pmatrix}$$

$$M \qquad\qquad\qquad\qquad M'$$

Figure 1. $R = P + (n-1)\,Q$.

In M', subtract column 1 from column j for each $j \geq 2$. The resulting matrix is a (block) lower triangular matrix M'' whose entries in the main diagonal are $P + (n-1)Q$, $P-Q$, $P-Q, \dots, P-Q$. Therefore, $\det M = \det M' = \det M'' = \det(P+(n-1)Q)\,(\det(P-Q))^{n-1}$. ■

Corollary. *Let A be an $n \times n$ matrix whose diagonal elements are all equal to p and all other elements equal to q. Then* $\det A = (p+(n-1)q)(p-q)^{n-1}$.

Observe that $A(K_n)$ satisfies the conditions in the Corollary with $p=0$ and $q=1$. Therefore, $\det A(K_n) = (-1)^{n-1}(n-1)$.

Theorem 1. *Let G be a graph. Then $G \times Kn$ is singular if and only if 1 or $1-n$ is an eigenvalue of $A(G)$.*

Proof: Let $|V(G)| = r$. Then $A(G \times K_n)$ is an $n \times n$ block matrix which contains only $A(G)$ in the diagonal and the $r \times r$ identity matrix I elsewhere. By Lemma 1, $\det A(G \times K_n) = \det(A(G)+(n-1)I)(\det(A(G)-I))^{n-1}$. Thus, $\det A(G \times K_n)=0$ if and only if 1 or $1-n$ is an eigenvalue of $A(G)$. ■

Theorem 2. *Let $n \geq 4$. Then $C_m \times K_n$ is singular if and only if $m \equiv 0 \pmod 6$.*

Proof: By Theorem 1, $C_m \times K_n$ is singular if and only if 1 or $1-n$ is an eigenvalue of $A(C_m)$. The eigenvalues [2] of $A(C_m)$ are

$$\lambda_\delta = w_\delta + w^{(m-1)\delta}, \quad \delta = 1, 2, \dots, m$$

where $w = \cos\frac{2\pi}{m} + i \sin\frac{2\pi}{m}$. Therefore, $\lambda_\delta = \cos\frac{2\pi\delta}{m} + i \sin\frac{2\pi\delta}{m} + \cos\frac{2\pi(m-1)\delta}{m} + i \sin\frac{2\pi(m-1)\delta}{m} = 2\cos\pi\delta \cos\frac{\pi(m-2)\delta}{m}$. It is clear that $|2\cos\pi\delta\cos\frac{\pi(m-2)\delta}{m}| \leq 2$ and hence, $\lambda^\delta \neq 1-n$. Therefore $C_m \times K_n$ is singular if and only if $\lambda_\delta = 1$ for some δ $1 \leq \delta \leq m$. We shall show that $\lambda_\delta = 1$ if and only if $m \equiv 0 \pmod 6$.

First, let $\lambda_\delta = 1$. Since $\cos\pi\delta = \pm 1$, we much have $\cos\frac{\pi(m-2)\delta}{m} = \pm \frac{1}{2}$. This implies that $\frac{\pi(m-2)\delta}{m} = \pm \frac{\pi}{3}\pi k$ for some integer k. Cancelling π and clearing of fractions, we have $3(m-2)\delta = \pm m + 3mk \Rightarrow m(3k - 3\delta \pm 1) + 6\delta = 0 \Rightarrow m \equiv 0$ (mod 6.)

Conversely, let $m \equiv 0 \pmod 6$. Taking $\delta = m/\delta$, we have $\lambda_\delta = 2\cos\pi\delta \times \cos\frac{\pi(m-2)\delta}{m} = 2\cos\pi\delta\cos(\pi\delta - \frac{\pi}{3}) = 1$. ■

Theorem 3. For $m \geq 1$, $n \geq 1$, $\det A(K_m \times K_n) = (-1)^{(m-1)(n-1)}(m-2)^{m-1}(n-2)^{m-1}(m+n-2)$.

Proof: Using Lemma 1 with $G = K_m$, we have
$\det A(K_m \times K_n) = \det (A(K_m) + (n-1(I (\det (A(K_m) - I))^{n-1}$.

By the Corollary to Lemma 1, $\det (A(K_m) + (n-1) I) = (n-1 + m-1)(n-2)^{m-1}$ and $\det (A(K_m) - I) = (-1 + m - 1)(-2)^{m-1}$. Therefore, $\det A(K_m \times K_n) = (-2)^{(m-1)(n-1)}(m-2)^{n-1}(n-2)^{m-1}(m+n-2)$. ■

Corollary. *$K_m \times K_n$ is singular if and only if $m = 2$, or $n = 2$,* or $m = n = 1$.

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