EFFECTS OF SUBDIVISION AND CONTRACTION OF EDGES ON THE DIMENSION OF A GRAPH

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ABSTRACT

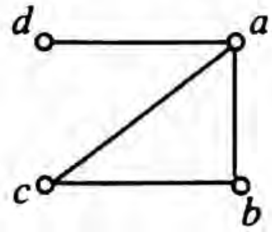
If the vertices of a graph can be associated bijectively with points in the *n*-dimensional Euclidean space E_n such that the distance between points associated with adjacent vertices is unity, then the graph is called a unit graph in E_n. The smallest n for which a graph G is a unit graph in E_n is called the dimension of G. Harary, et al, sometime in the 60's determined the dimension of some graphs and gave upper bounds for the dimension of a graph in terms of the number of vertices and in terms of the chromatic number. The effects of two graph operations on the dimension of a graph are considered here. An edge subdivision means inserting one new vertex in an edge of a graph. An edge contraction means reducing an edge to a single vertex by identifying its end vertices. Here, we show that the edge subdivision or edge contraction may either increase, decrease or leave the dimension of a graph unchanged. We prove here that every graph with n vertices and m edges can be subjected to a finite number of edge subdivisions to obtain a unit graph in E_2 with n+m vertices and 2m edges. Likewise, a Hamiltonian graph with n vertices and m edges can be subjected to a finite number of edge subdivisions to yield a unit graph in E_2 with m vertices and $2m - n$ edges. Most results are proven by actual construction.

Key words: Euclidian space, distance, dimension, graph, edge subdivision, edge contraction, Hamiltonian

1. INTRODUCTION

By a *graph we* shall understand a finite, loopless graph without multiple edges. If G is a graph, we shall denote by $V(G)$ the set of vertices of G, and by $E(G)$ the set of its edges. We shall write $G = \langle V(G), E(G) \rangle$ An edge joining x and y shall be denoted by $|x,y|$.

Example. Let G be the graph defined by $V(G) = \{a,b,c,d\}$, and $E(G) = \{[a, b], [b, c], [c, a],[a, d]\}$. We represent G pictorially as follows:

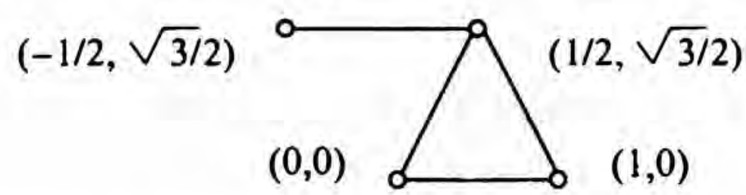


The total number of vertices in a graph is called its *order*. The *size* of a graph is the total number of edges in it. One graph which is of importance in this study is the *complete graph of order n,* denoted by K_n. The readers may please refer to (4) for other terms and concepts whose definitions are not given here.

Let n be a positive integer. The set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers x_i will be denoted by E_n. The elements of E_n will be called *points*. If $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$ are two points in E_n, we define their sum as $p + q = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. If c is any real number, we define $cp = (cx_1, cx_2, \ldots, cx_n)$. Under these operations, E_n is a *vector space of dimension n*. Thus, we could also call the elements of E_n as vectors instead of points. We further define the *distance* between p and q by $d(p,q) = \{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2\}^{1/2}$. We shall refer to E_n as the *n-dimensional Euclidean space,* or the *Euclidean n-space.* For convenience, the 0-dimensional Euclidean space E_0 will be understood to be the trivial vector space, i.e., the vector space containing only the zero vector.

Definition 1 Let G be a graph with vertices $u_1, u_2, \ldots, u_n$ If for some $k \geq 0$, there is one-to-one mapping f: $V(G) \to E_k$ such that the distance between $f(u_i)$ and $f(u_j)$ is 1 whenever u_i and u_j are adjacent, then we shall call f a *unit representation* of G in E_k. We shall call G a *unit graph* in E_k if G has a unit representation in E_k. The smallest k for which G is a unit graph in E_k is called the *dimension* of G, written as dim G.

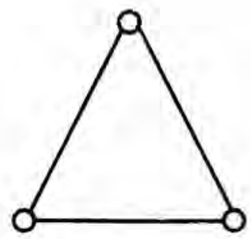
Example. The graph in the last example is a unit graph in E_2 and a unit representation of G in E_2 is shown below.



$$\phi: \begin{cases} a \to (-1/2, \sqrt{3}/2) \\ b \to (1, 0) \\ c \to (0, 0) \\ d \to (-1/2, \sqrt{3}/2) \end{cases}$$

It is quite obvious that the graph in the above example has no unit representation in E_1.Thus, it has dimension 2.

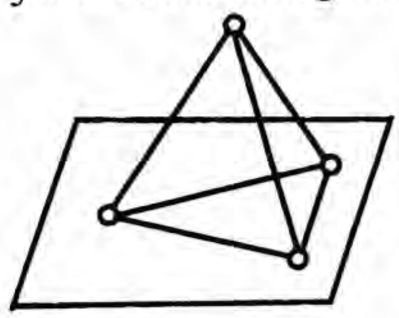
One graph of special importance is the complete graph of order n. This graph consists of n vertices which are pairwise adjacent. For example, K_3 can be described as the graph one of whose pictorial representations looks like a triangle.



A unit representation of K_3 in E_2.

It is quite obvious that K_3 which has a unit representation in E_2 has no unit representation in E_1. Thus, dim $K_3 = 2$.

Forming a unit representation of K_n may be described as follows: Given $\binom{n}{2}$ sticks, each one unit long, join the sticks at their ends to produce the maximum number $\binom{n}{3}$ of congruent equilateral triangles. The solution in the case of three sticks is shown in the preceding figure. For $n = 4$, we are given $\binom{4}{2} = 6$ sticks. Therefore, we need to add three more sticks in the triangle in the last figure to form a total of $\binom{4}{3} = 4$ congruent equilateral triangles. It is easy to see that this has no solution in the plane. Thus, we are forced to go to a higher dimension. A unit representation of K_4 in E_3 is shown in the figure below.



A representation of K_4 in E_3.

One basic question is whether every graph is a unit graph in some Euclidean n-space. This is answered by the corollary to the following theorem.

2. KNOWN RESULTS AND PRELIMINARY CONCEPTS

Therorem 1 (1), (2), (3) If K_n *is the complete graph of order* $n \geq 1$, then dim $K_n = n - 1$.

Since every graph of order n is a spanning subgraph of K_n, the following corollary immediately following corollary immediately follows:

Corollary 1 If *G is any graph of order n, then* dim $G \geq n - 1$.

Theorem 2 (1), (2), (3)) *The complete bipartite graph* $K_{m,n}$ *has dimension given by the following:*

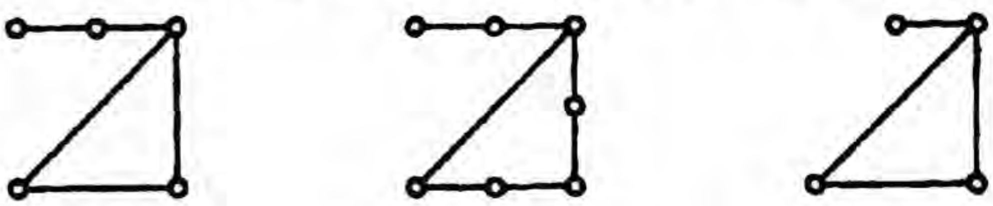
$$\dim K_{m,n} = \begin{cases} 1 & \text{if } m = 1 \text{ and } n = 1 \text{ or } 2 \\ 2 & \text{if } m = 1 \text{ and } n \geq 3 \\ 2 & \text{if } m = 2 \text{ and } n = 2 \\ 3 & \text{if } m = 2 \text{ and } n \geq 3 \\ 4 & \text{if } m \geq 3 \text{ and } n \geq 3 \end{cases}$$

Let us now define two operations on graphs, namely *subdivision* and *contraction*.

Definition 2 Let G be a graph. To *subdivide* an edge $[x, y]$ of G means to remove the edge $[x, y]$ and add a new vertex z and two new edges $[x, z]$ and $[z, y]$. A *subdivision* of G is any graph obtained from G by a finite sequence of edge subdivisions.

Definition 3 Let G be a graph and let $[x, y]$ be an edge of G such that x and y are not adjacent to a common vertex. To contract the edge $[x, y]$ means to remove the edge $[x, y]$ and to identify x and y. A *contraction* of G is any graph obtained from G by a finite sequence of edge contractions.

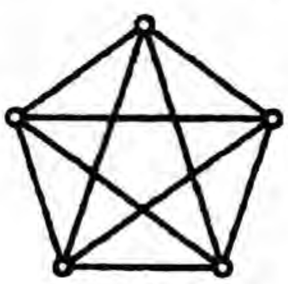
Example. In the figure below, G' is a subdivision of G while G'' is a contraction of G.



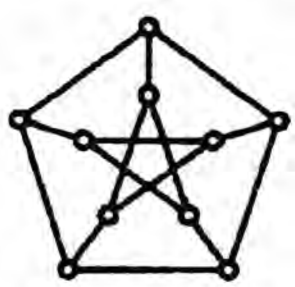
G G' G''

Definition 4 If H is a graph obtained from G by applying a finite sequence of operations consisting of subdivisions and contractions, we call H a *home-omorph* of G. Two graphs G_1 and G_2 are said to be *homeomorphic* if they have respective homeomorphs H_1 and H_2 which are isomorphic.

Example. The complete graph K_5 and the Petersen graph shown below are homeomorphic.



K_5



Petersen graph

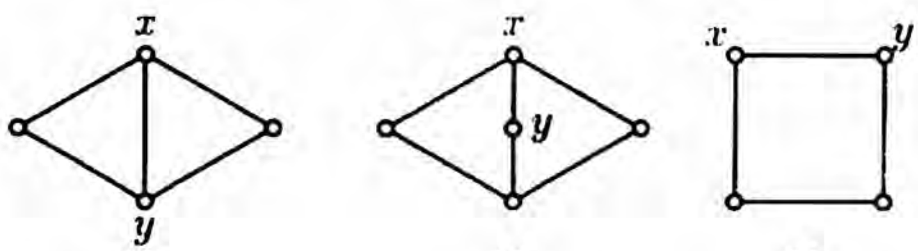
Note that a sequence of 5 contractions will transform the Petersen graph into the complete graph K_5. No edge of K_5 may be contracted and hence it is a full contraction of the Petersen graph. This is the case where full contraction increases the dimension of the graph. The Petersen graph is of dimension 2 while the complete graph K_5 has dimension 4.

3. MAIN RESULTS

Our next two theorems give the general effects of edge subdivision and edge contraction on the dimension of a graph.

Theorem 3 *An edge subdivision may either increase, decrease, or not change the dimension of a graph.*

Proof: Consider the graphs G_1 , G_2, and G_3 shown below.

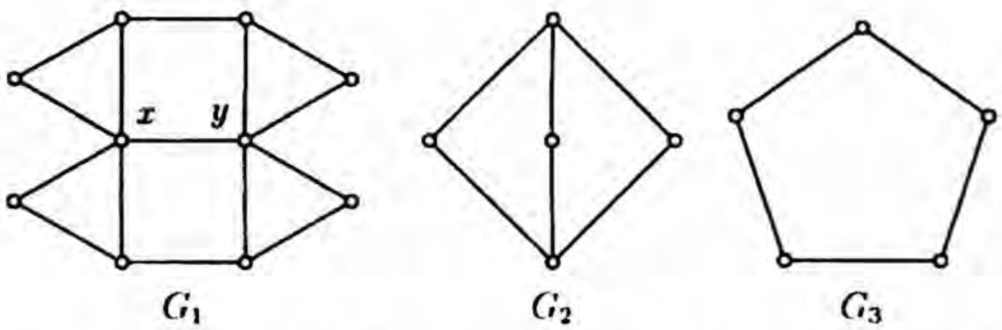


G_1 G_2 G_3

It is easy to see that dim $G_1 = 2$, dim $G_2 = 3$ and dim $G_3 = 2$. By subdividing the edge $[x, y]$ in each graph, we obtain graphs G'_1, G'_2 and G'_3. It is easy to check that dim $G'_1 = 3$, dim $G'_2 = 2$ and dim $G'_3 = 2$. Thus, in the first case, there is an increase in dimension. In the second case, there is a decrease in dimension. In the last case, there is no change in dimension. □

Theorem 4 *An edge contraction may either increase, decrease or not change the dimension of a graph.*

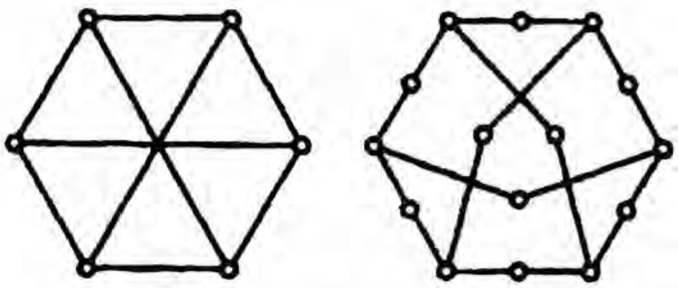
Proof: Consider the graphs G_1, G_2, and G_3 shown below.



G_1 G_2 G_3

For the graph G_1, we have dim $G_1 = 2$. By contracting the edge $[x, y]$, we get a graph whose dimension is 3. If any edge of G_2 is contracted, the dimension will change from 3 to 2. If any edge of G_3 is contracted, the dimension remains constant at 2.

Suppose each edge of a graph is subdivided, what happens to the dimension? Below, we show a graph G and the graph H obtained from G by subdividing every edge of G.



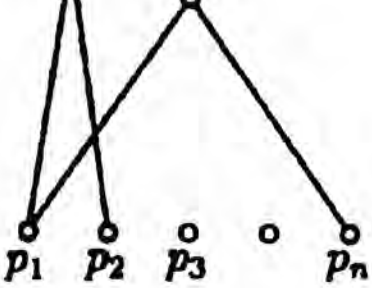
G H

The graph G is seen to be $K_{3,3}$ and has dimension 4. We shall see later that the graph H, which is a full subdivision of G, is of dimension 2.

If we subdivide all the edges of a given graph, the result is a bipartite graph. In view of Theorem 2, the dimension of this bipartite graph is at most 4. The next theorem gives a better estimate of the dimension of the full subdivision graph of a graph.

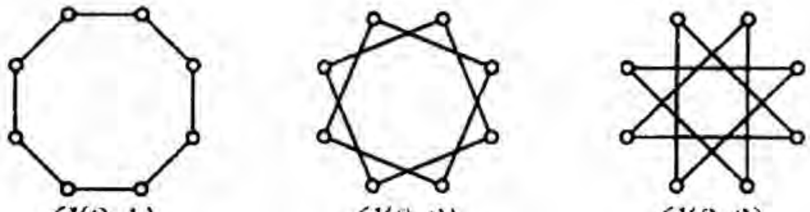
Theorem 5 *Let G be a graph of order n and size m. There exists a subdivision graph H of G of order $n+m$ and size $2m$ such that H is a unit graph in E_2.*

Proof: Let $u_1, u_2, \ldots, u_n$ be the vertices of G and associate them with collinear points $p_1, p_2, \ldots, p_n$ in E_2 such that the distance between p_i and p_{i+1} is less than $1/n$. (Please refer to the figure below.) Then p_1 and p_n are farthest apart with a distance of less than $(n-1)/n < 1$ from each other. Whenever u_i and u_j are adjacent, we introduce a new point p_{ij} in E_2 which is 1 unit away from both p_i and p_j. The total number of new points we have to add is clearly equal to m. Corresponding to p_{ij}, we subdivide the edge $[u_i, u_j]$ and introduce the subdivision vertex u_{ij}. The resulting subdivision graph of G is clearly of order $n+m$ and has a unit representation in E_2.



Definition 5 Let $n \geq 3$ and let $1 \leq k < n/2$. We define the graph $G(n, k)$ to be the graph with vertices $1, 2, \ldots, n$ whose edges are $[i, i+k]$, where $k = 1, 2, \ldots, n$. The sum $i + k$ is to be read modulo k.

Some examples of graphs $G(n, k)$ are given below.



$G(8,1)$ $G(8,2)$ $G(8,3)$

Lemma 1 *Let $n \geq 3$ and let $1 \leq k < n/2$. Then $G(n, k)$ is a cycle if and only if n and k are relatively prime.*

Proof: First, assume that n and k are relatively prime. We claim that the sequence $[1, 1 + k, 1 + 2k, \ldots, 1 + (n - 1)k]$ is a path in $G(n, k)$. The elements of the sequence are to be read modulo n. It is clear that consecutive vertices in the sequence are edges of G, by definition. Suppose that two vertices in the path are equal, say $1 + ik \equiv 1 + jk \pmod{n}$. Then $ik \equiv jk \pmod{n}$, and $i \equiv j \pmod{n}$ since n and k are relatively prime. But $0 \leq i \leq n - 1$ and $0 \leq j \leq n - 1$. It follows that $i = j$. Now, the last vertex of the path is adjacent to the first vertex 1 because $1 + (n - 1)k + k = 1 + nk \equiv 1 \pmod{n}$. Therfore, $G(n, k)$ is a cycle.

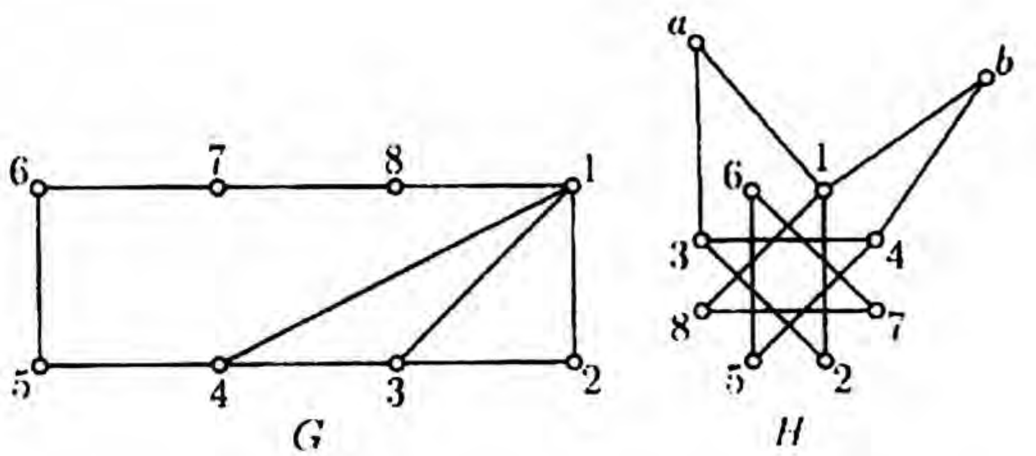
Next, let us assume that n and k are not relatively prime and let their greatest common divisor be equal to $d > 1$. Let $n' = n/d$ and consider the sequence $[1, 1 + k, 1 + 2k, \ldots, 1 + (n' - 1)k]$. Consecutive vertices of this sequence are adjacent by definition of G. If $1 + ik \equiv 1 + jk \pmod{n}$, we would have $ik \equiv jk \pmod{n}$. If we divide the congruence through by k, we get $i \equiv j \pmod{n'}$. But each of i and j ranges from 0 to $n' - 1$ only. It follows that $i = j$. The last vertex of the path is adjacent to the first vertex 1since $1 + (n' - 1)k + k = 1 + n'k = 1 + n\left(\frac{k}{d}\right) \equiv 1 \pmod{n}$. Therefore, $G(n, k)$ contains as a subgraph a cycle of length n' which is less than n. Consequently, $G(n, k)$ is not a cycle. □

Theorem 6 *Let G be a Hamiltonian graph of order $n \neq 4$ or 6, and size m. Then G has a subdivision graph H of order m and size $2m - n$ such that H is a unit graph in E_2.*

Proof: Let G be a Hamiltonian graph of order $n \neq 4$ or 6. Let us first consider the case $n = 5$. Consider a regular pentagon in E_2 whose sides have length equal to 1 unit. Let p_1, p_2, p_3, p_4, and p_5 be consecutive vertices of the polygon. If $C = [u_1, u_2, u_3, u_4, u_5, u_1]$ is a Hamiltonian cycle in G, let the vertices u_1, u_2, u_3, u_4, u_5 correspond to the consecutive vertices of the Hamiltonian cycle C which are adjacent, then subdivide the edge $[u_i, u_j]$ using a vertex u_{ij}. Let u_{ij} be associated with a point in E_2 which is 1 unit away from both p_i and p_j. Clearly, such a point, say p_{ij} (not a vertex of the pentagon) is uniquely found in E_2. Repeat this process for every pair of adjacent but non-consecutive vertices of C. The result is a subdivision of G which is of order m and size $2m - 5$ which has a unit representation in E_2. □

Let G be a Hamiltonian graph of order $n > 6$. If n is even, say $n = 2^r m$, where m is odd, choose $k = 2^{r-1}m - 1$. Then n and k are relatively prime. Draw the graph $G(n,k)$ such that the edges are 1 unit long. If $[x, y]$ is an edge of G such that x and y are not adjacent in $G(n,k)$, we subdivide $[x, y]$ into two edges $[x, z]$ and $[z, y]$, each one unit long. Do this for all other similar edges. The result is a subdivision graph of H of G which is a unit graph in E_2. Furthermore, the order of H is $n + m - n = m$ and its size is $n + 2(m - n) = 2m - n$.

Example. We illustrate in the figure below how to subdivide a Hamiltonian graph of order 8 so that the resulting graph has a unit representation in E_2.



In the above figure, we are given a Hamiltonian graph G with spanning cycle $[1,2,3,4,5,6,7,8]$. The new vertices a and b are introduced to subdivide the edges $[1,3]$ and $[1,4]$ respectively. The graph H is the result of the edge subdivisions and a unit representation of H is shown in the same figure.

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