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COMMUTATIVE CENTRALIZER ALGEBRAS AND RELATED STRUCTURES[1]

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ABSTRACT

Let $M(x)$ be a completely reducible matrix representation of a group G over a field K. The set $C(M)$ of matrices T over K such that $TM(x) = M(x)T$ for all $x \in G$ forms an algebra over K called the centralizer algebra of M. In this paper we investigate the centralizer algebras of permutation representations. We prove a sufficiency condition for the commutativity of centralizer algebras of semi-direct products and also obtain several commutativity results coming from some classes of finite groups using character theory and other techniques. Finally we show how these algebras arise in various contexts and discuss some related structures.

Keywords: *representation, centralizer algeba, Hecke algebra, group algebra, Bose-Mesner algebra, permutation character, symmetric group.*

1. INTRODUCTION

One of the most fundamental results in representation theory is Schur's Lemma which says that if $M(x)$ is an irreducible matrix representation of a group G over an algebraically closed field, then the only matrices which commute with all the matrices $M(x)$, $x \in G$, are the scalar mutiples of the unit matrix.

It is interesting to find that the converse is also true, that is, if the only matrices that commute with $M(x)$ are the scalar multiples of the unit matrix, then $M(x)$ is irreducible. This result is part of a more general situation.

Let $M(x)$ be a completely reducible matrix representaion of a group G over

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a field K. The set $C(M)$ of all matrices T over K satisfying $TM(x) = M(x)T$ for all $x \in G$ forms an algebra over K, called the *centralizer algebra* of M.

We recall that an algrebra is a vector space which also an (associative) ring in the usual way. It is a subalgebra of the complete matrix algebra $\mathcal{M}_m(K)$, where m is the degree of M. It is not difficult to prove that if A and B are equivalent representations, then their corresponding centralizer algebras are isomorphic.

In this paper we investigate centralizer algebras of transitive permutation representations and prove several commutativity results. Commutativity is a much desired condition that facilitates the investigation of the algebra and its representation theory. We shall consider a more general structure called a Hecke algebra and introduce the notion of a Gelfand pair. We conclude by discussing related structures arising from these algebras.

2. HECKE ALGEBRAS

In this section we shall consider a more general structure called a Hecke algebra, a particular case of which is our previously defined centralizer algebra. let us start with a brief historical background.

In 1937 E. Hecke introduced certain endomorphisms of the space of entire elliptic modular forms, now called Hecke operators (Hecke, 1937). About two decades later, G. Shimura defined an abstract Hecke algebra that gave an algebraic setting for different kinds of Hecke operators (Shimura, 1959).

Shimura's construction involved a group G and a subgroup H of G. Letting $H\backslash G/H = \{HgH : g \in G\}$ denote the set of (H, H)-double cosets of G, a product of two double cosets is defined as an integral linear combination of all double cosets. The abstract Hecke algrebra, $\mathcal{H}(G, H)$ is the free Z-module with $\{HgH\}$ as a basis. If H is a normal subgroup of G, $\mathcal{H}(G, H) \cong Z[G/H]$.

In the 1960s important work was done by mathematicians such as C.W. Curtis and N. Iwahori who considered Chevalley groups and various distinguished subgroups. They defined Hecke algebras in terms of generators and relations and used the notion of generic algebras. See (Curtis, Iwahori, Kiloyer, 1971) and (Curtis, Reiner, 1981). Currently there is much interest in cyclotomic Hecke algebras which treat complex reflection groups (Malle, 1998). The study of Hecke algebras continues to this day. The theory has a wide range of applications from number theory to representation theory as well as the structure theory of finite groups and other algebraic objects.

Today there are several closely related definitions of Hecke algebras, not all equivalent. We give a particularly simple one. Recall that an idempotent in an algebra $\mathcal{R}$ is a nonzero element $e \in \mathcal{R}$ such tha $e^2 = e$.

Definition 1. *(Hecke algebra)*

The subalgebra $e\mathcal{R}e = \{ere : r \in \mathcal{R}\}$ of $\mathcal{R}$ is called a *Hecke algebra*. The identity of $e\mathcal{R}e$ is the idempotent e.

Remarks.

1. Some authors refer to these algebras as *Iwahori-Hecke* algebras in honor of Iwahori, who did much of the pionering work in this area.
2. One point of the study of these algebras is that $e\mathcal{R}e$ is often easier to understand than the algebra $\mathcal{R}$.

Example 1.

Let K be a field and $M_n(K)$ be the algebra of $n \times n$ matrices, with $1 \le m \le n$. Let e be the $n \times n$ matrix whose i, j-entry is 1 if $1 \le i = j \le m$ and 0 otherwise. Then $eM_n(K)e$ consists of the matrices whose i, j-entry equals 0 unless both i and j lie between 1 and m. Hence, $eM_n(K)e \cong M_m(K)$.

To prepare for our next examples, we need to review the concept of group algebras. In this paper, we consider only finite groups.

Example 2.

Let $G = \{g_i : i \in I\}$ be any finite group, and R any commutative ring with unity. Let RG be the set of all formal sums $\Sigma_{i\in I}\alpha_i g_i$, $\alpha_i \in R$ and $g_i \in G$, where all but a finite number of the α_i are 0. We define equality of formal sums if and only if all coefficients are equal.

Next, addition in RG is defined as:

$$\sum_{i\in I}\alpha_i g_i + \sum_{i\in I}\beta_i g_i = \sum_{i\in I}(\alpha_i + \beta_i)g_i;$$

and multiplication by:

$$\left(\sum_{i\in I}\alpha_i g_i\right)\left(\sum_{i\in I}\beta_i g_i\right) = \sum_{i\in I}\left(\sum_{g_j g_k = g_i}\alpha_j \beta_k\right)g_i .$$

In the above product, we formally distribute the sum $\Sigma_{i\in I}\,\alpha_i g_i$ over the sum $\Sigma_{i\in I}\,\beta_i g_i$ and rename a term $\alpha_j g_j \beta_k g_k$ by $\alpha_j\beta_k g_i$, where $g_j g_k = g_i$.

Definition 2. *(group ring, group algebra)*

Under the operations above, $(RG, +, .)$ is a ring, called the *group ring* of G over R. If k is a field, the kG is called the *group algebra*.

We shall focus on group algebras kG, where k is a field of characteristic zero. In this case, scalar multiplication is difined by

$$\alpha\left(\sum_{i\in I}\alpha_i g_i\right) = \sum_{i\in I}\alpha\alpha_i g_i,\ \alpha \in k .$$

The formal sum $1 \cdot e$, $1 \in K$, $e \in G$ is the identity in kG. The set of formal sums $\{g^* = 1 \cdot g, g \in G\}$ is linearly independent and is a basis of kG over k. The map $g \to g^*$ is an isomorphism of G into kG, and we will identify G with its image under this isomorphism.

Suppose that H is a subgroup of G. The element $e = \frac{1}{|H|}\Sigma_{h\in H}\, h$ of the group algebra kG is clearly an idempotent. So that if $H = \{1\}$,then $ekGe = kG$ and if $H = G$, we have $ekGe = \{\alpha e : \alpha \in k\}$.

We can identify kG with the set of all k -valued functions on G, where

$$\sum_{g\in G} \alpha_g g \longrightarrow f: G \to k,$$

where the function f maps the group elements g to α_g. Under the correspondence, addition of functions in G is defined by

$$(f_1 + f_2)(g) = f_1(g) + f_2(g).$$

As for multiplication,

$$(\sum_{x\in G}\alpha_x x)\ (\sum_{y\in G}\beta_y y) = \sum_{z\in G}\ (\sum_{xy=z}\alpha_x \beta_y) z = \sum_{z\in G}\gamma_z z,$$

where

$$\gamma_z = \Sigma\alpha_x\beta_{x^{-1}z}.$$

So, multiplication of functions is defined by convolution, i.e. $f_1 f_2 = f_3$, where

$$f_3(z) = \sum_{x\in G} f_1(x)\, f_2(x^{-1}z).$$

Under the above identification of kG with $\{f : G \to k\}$, we see that we can identify the Hecke algebra as follows:

$$ekGe = \{f : G \longrightarrow k\} : f(h_1 g h_2) = f(g)\ \forall h_1, h_2 \in H\}.$$

Thus the Hecke algebra $ekGe$, where $e = \frac{1}{|H|}\Sigma_{h\in H}\, h$ is the subalgebra of k-valued functions that are constant on the (H, H)-double cosets of G.

Remark.

In the above construction, if one takes G to be a transitive permutation group and H a subgroup stabilizing a point, then the Hecke algebra $\mathcal{H}(G, H)$ is naturally isomorphic to the centralizer algebra of the permutation representation.

Nontrivial examples are obtained by taking particular proper subgroups of G. See (Curtis, Iwahori, Kilmoyer, 1971) and its references. For instance, a deep result shows that if $G = GL(n, q)$, the group of all invertible $n \times n$ matrices over a finite field of order q and H is the subgroup of all lower triangular matrices, then $e\,kGe \cong kS_n$, the group algebra of the symmetric group of degree n.

We can generalize the above construction in the following way.

Example 3.

Let ψ be an irreducible character of a subgroup H of G. Then the element

$$e = \frac{\psi(1)}{|H|} \sum_{h \in H} \psi(h^{-1})\, h$$

is an idempotent. One can define the Hecke algebra $H = ekGe$. Our previous example is the special case where ψ is the trivial character.

This construction is important for the representaion theory G. We recall that every character X of G can be regarded as a character of kG. Also if X is an irreducible character of kG, the restriction of X to $\mathcal{H}$ is either zero or is an irreducible character of $\mathcal{H}$. Furthermore, every irreducible character of $\mathcal{H}$ is the restriction of some irreducible character of kG.

The reader may consult the book (Curtis, Reiner, 1981) for more details of the Hecke algebra and representation theory.

3. COMMUTATIVITY AND GELFAND PAIRS

In this section we consider the centralizer or Hecke algebra of k-valued functions on G that are constant on the (H, H)-double cosets of G defined in Example 2, and prove several results concerning commutativity of these algebras. As before let us denote these algebras by $\mathcal{H}(G, H)$. We have the following definition.

Definition 3. *Gelfand pair*

If $\mathcal{H}(G, H)$ is commutative, we call (G, H) a *Gelfand pair*.

The terminology comes from the work of (I.M. Gelfand, 1950) on Lie groups in the 1950s who proved that if G is a Lie group and H is a compact subgroup of G, the algebra of all integrable functions $\phi(g)$ satisfying the condition that $\phi(h_1 g h_2) = \phi(g)$ for almost all g and all $h_1, h_2 \in H$ is commutative. In this algebra, the product of two functions is given by convolution, i.e., $\phi = \phi_1 \times \phi_2$, where

$$\phi(g) = \int \phi(gg_1^{-1})\phi_2(g_1)dg_1,$$

and addition is as usual.

For our first results, we will use the following proposition proved in (Balmaceda, 1996).

Recall that an anti-automorphism of a group G is a bijection ϕ from G onto itself with $\phi(g_1 g_2) = \phi(g_2)\phi(g_1)$ for all $g_1, g_2 \in G$.

Proposition 1. (Balmaceda, 1996)

Let G be a finite group and $H \leq G$. Suppose there exists an anti-automorphism σ of G satisfying $(HgH)^\sigma = HgH$ for all $g \in G$. Then $\mathcal{H}(G, H)$ is commutative.

W now use this result to obtain the next two theorems.

Theorem 2.

Let K be a finite group and put $G = K \times K$. Let $H = \{(a, a) | a \in K\}$. Then (G, H) is a Gelfand pair.

Proof. One can check that the map σ, where $(a, b)^\sigma := (b^{-1}, a^{-1})$ is an anti-automorphism of G fixing every double coset.

The above result can also be proved using character theory. However, the proof given above is probably the shortest. In the next application we prove a result on semi-direct products of groups. We recall its definition.

Definition 4. *Semi-direct Product*

Let two groups H and N be given as well as a homomorphism τ : H $\to$ Aut$N$, $H \to \tau_h$. We define a product on $N \times H$ via

$$(n, h) \cdot (n_1, h_1) := (n\tau_h(n_1), hh_1).$$

Thus we obtain a group structure on $N \times H$, the semi-direct product of $N \times H$, which is denoted by $N \times_\tau H$.

Due to the embedding $N \to N \times_\tau H$, $n \to (n, e)$, we can regard N as a normal subgroup of $N \times_\tau H$. Moreover, H can be embedded into $N \times_\tau H$ via $H \to N \times_\tau H$, $h \to (e, h)$.

Theorem 3.

Let $G = N \times_\tau H$ be the semi-direct product of N and H. Then (G, H) is a Gelfand pair, whenever N is abelian.

Proof. From the definition of the product, each right coset relative to H contains a unique representative of the form (n, e), $n \in N$. A straightforward calculation shows that the double coset $H(n, e)H$, $n \in N$ is the disjoint union of the right costs $H(m, e)$, $m \in \{\tau_h(n) | h \in H\}$.

Suppose now that N is abelian and consider the map $\phi : G \to G$, where

$$(n, h) \to (\tau_h^{-1}(n), h^{-1}) = (n^{-1}, h)^{-1}.$$

Direct calculation shows that ϕ is an anti-automorphism of G satisfying $\phi(H) = H$ and $(H(n, e)\, H)^{\phi} = H(n, e)\, H$. An application of Proposition 1 completes the proof.

The above theorem can be applied to any semi-direct product. For example, one can take the dihedral groups D_n of order 2_n defined by

$$D_n = < a, b \mid a^n = b^2 = e,\ bab = a^{-1} >.$$

Then D_n is the semi-direct product of the cyclic groups generated by a and b. If one takes $H = < b > = \{e, b\}$, then using the above theorem, we see that $\mathcal{H}(D_n, H)$ is commutative.

In representation theory, the study of Hecke algebras arises in the investigation of induced representations and their irreducible decompositions. The following theorem is known (Curtis, Riener, 1981).

Proposition 5.

Let G be a transitive permutation group and H be the stabilizer in G of a point. then $\mathcal{H}(G, H)$ is commutative if and only if the permutation character is a sum of distinct irreducible characters.

If π is the permutation character of G, then $\pi(g)$ is the number of points fixed by the action of the element g of G. Furthermore if G is transitive, we know that $\pi = (1_H)^G$, the character of G induced from the trivial character of H. Hence to determine if (G, H) forms a Gelfand pair, we need to know the decomposition of $(1_H)^G$ into irreducible characters of G.

Out last results in this paper involve Gelfand pairs in wreath product subgroups of symmetric groups. We will use the approach using permutation characters. Before we state and prove the next theorem, we define the following. Let n be any even integer, say $n = 2k$. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of the integer k, then $(2\lambda_1, \ldots, 2\lambda_r)$ is a partition of n. We denote this partition of n by 2λ and call 2λ an even partition.

Consider the subgroup $S_2 \wr S^k$ of the symmetric group S_{2k}. Let

$$\pi_{2,k} = (1_{S2 \wr Sk})^{S2k}.$$

We will show that the irreducible constituents of the above permutation character are distinct. To do this we need a lemma.

Lemma 6.

$$(\pi_{2,k})_{S2k-1} = (\pi_{2,k-1})^{S2k-1}.$$

Proof. By Mckey's subgroup theorem (Curtis, Renier, 1981), the right side equals.

$$\sum_{t} (1_{S_2 \wr S_k \cap (S_{2k-1})^t})^{S_{2k-1}},$$

the sum running over all representatives $(S_2 \wr S_k)tS_{2k-1}$ of double cosets. As $S_2 \wr S_k$ is transitive, there is exactly one such double coset, namely $(S_2 \wr S_k)\ S_{2k-1}$ and furthermore $S_2 \wr S_k \cap S_{2k-1} = S_2 \wr S_{k-1}$ since a one-point stabilizer will fix the block containing that point, and will act as $S_2 \wr S_{k-1}$ on the remaining 2k – 2 points. Thus we can proceed as follows.

$$(\pi_{2,k})_{S_{2k-1})} = (1_{S_2 \wr S_{k-1})})^{S_{2k-1}}.$$

By transitivity of induction,

$$(\pi_{2,k})_{S_{2k-1})} = ((1_{S_2 \wr S_{k-1})})^{S_{2k-2}})^{S_{2k-1}}.$$

By definition of $\pi_{2,k-1}$, this gives the result

$$(\pi_{2,k})_{S_{2k-1})} = (\pi_{2,\,k-1)})^{S_{2k-1}}. \square$$

Theorem 7. *If n is an integer, $n = 2k$, then the character $\pi_{2,k}$ decomposes as*

$$\pi_{2,k} = \sum_{\lambda \vdash k} \chi^{2\lambda},$$

where the sum runs over all partitions of the integer k.

Proof. We prove the theorem by induction on k. The case $k = 1$ is obvious. If $k > 1$, by the inductive hypothesis, $\pi_{2,k-1}$ is the sum of distinct irreducible characters of S_{2k-2} corresponding to even partitions of $2k - 2$, each occuring exactly once.

By the Branching Rule $(\pi_{2,\,k-1)})^{S_{2k-1}}$ is the sum of all the irreducible characters of S_{2k-1} corresponding to partitions of $2k - 1$ with exactly one odd part. So by the proceeding lemma, we get that $(\pi_{2,k})_{S_{2k-1})}$ is equal $\Sigma_\beta \chi^\beta$, where β runs through the partitions of $2k - 1$ with exactly one odd part.

Another appliction of the Branching Rule shows that for these β, the following is true:

$$\sum_{\beta} \chi^{\beta} = \sum_{\lambda \vdash k} (\chi^{2\alpha})^{S_{2k-1}}.$$

It therefore suffices to prove that $\pi_{2,k}$ cannot contain any connstituent $\mathcal{X}^{\gamma}$ with odds parts γ_i. Since for each such γ, every constituents of $(\mathcal{X}^{\alpha})_{S2k-1}$ has exactly one odd part, γ must be of the form $\gamma = (\gamma_1, \gamma_2)$, with γ_i odd. Hence it remains to show that no $\gamma = (\gamma_1, \gamma_2)$, γ_i odd, can be a constituent of $\pi_{2,k}$.

We notice that the induction hypothesis and the previous lemma yield that $\pi_{2,k}$ decomposes into distinct irreducible characters. Now $\mathcal{X}^{(2k)}$ is a constituent of $\pi_{2,k}$ since $\pi_{2,k} = (1_{S2 \wr Sk})^{S2k}$. We need to show that $\mathcal{X}^{(2k-1,1)}$ cannot occur in the decomposition. As

$$(\mathcal{X}^{(2k-1,1)})_{S2k-1} = \mathcal{X}^{(2k-2,1)} + \mathcal{X}^{(2k-1)},$$

the occurence of both $\mathcal{X}^{(2k)}$ and $\mathcal{X}^{(2k-1,1)}$ in $\pi_{2,k}$ would imply that $\mathcal{X}^{(2k-1)}$occurs twice in $(\pi_{2,k-1})^{S2k-1}$ which is in fact a sum of distinct irreducibles. Thus, $\mathcal{X}^{(2k)}$ is a contituent of $\pi_{2,k}$ while $\mathcal{X}^{(2k-1,1)}$ is not.

In order to prove that $\mathcal{X}^{(2k-2,2)}$ is a constituent of $\pi_{2,k}$ we consider $\mathcal{X}^{(2k-2,1)}$, which is a constituent of $(\pi_{2,k-1})^{S2k-1}$. It satisfies

$$(\mathcal{X}^{(2k-2,1)})^{S2k} = \mathcal{X}^{(2k-1,1)} + \mathcal{X}^{(2k-2,2)} + \mathcal{X}^{(2k-2,1,1)}.$$

Hence $\mathcal{X}^{(2k-2,1)}$ arises from the restriction of $+ \mathcal{X}^{(2k-2,2)}$, for we have seen already that neither $\mathcal{X}^{(2k-1,1)}$ nor $\mathcal{X}^{(2k-2,1,1)}$ occur in $\pi_{2,k}$.

This suggests how we can proceed inductively. Let us assume that we have shown that $\mathcal{X}^{(2k-2,l)}$, l even, is a constituent of $\pi_{2,k}$. We need to check that $\mathcal{X}^{(2k-l-1,\, l+l)}$ does not occur, while $\mathcal{X}^{(2k-l-2,\, l+2)}$ does. As

$$(\mathcal{X}^{(2k-l-1,\, l+1)})_{S2k-1} = \mathcal{X}^{(2k-l-2,\, l+l)} + \mathcal{X}^{(2k-l-1,1)},$$

and

$$(\mathcal{X}^{(2k-l,l)})_{S2k-1} = \mathcal{X}^{(2k-l-1,\, l)} + \mathcal{X}^{(2k-l,l-1)}.$$

then $\mathcal{X}^{(2k-l-1,\, l+1)}$ cannot occur, for otherwise, $(\pi_{2,k-1})^{S2k-1}$ would have repeated constituents. On the other hand, $\mathcal{X}^{(2k-l-2,\, l+1)}$ is a contituent of $(\pi_{2,k-1})^{S2k-1}$ and

$$(\mathcal{X}^{(2k-l-2,\, l+1)})^{S2k} = \mathcal{X}^{(2k-l-1,\, l+1)} + \mathcal{X}^{(2k-l-2,l+2)} + \mathcal{X}^{(2k-l-2,l+1,1)},$$

so that $\mathcal{X}^{(2k-l-2,\, l+1)}$ arises from the restriction of $\mathcal{X}^{(2k-l-2,\, l+2)}$ which therefore must occur in $\pi_{2,k}$. This completes the proof.

Corollary 8.

$(S_{2k}, S_2 \wr S_k)$ *is a Gelfand pair.*

Remarks.

If we consider the analogous situation for the alternating group, we find that the pair $(A_{2k}, (S_2 \wr S_k) \cap A_{2k})$ does not always give a Gelfand pair. to see this, we first note that by Mackey's theorem and the preceding theorem, the corresponding permutation character is given by

$$(\pi_{2,k})_{A2k} = \sum_{\lambda \vdash k} (\chi^{2\lambda})_{A2k}.$$

If $k \equiv 0 (mod 4)$, say $k = 4n$ for some positive integer, n, then the partitions $(2n, 2n, 2n, 2n)$ and (4^{2n}) are both even partitions of $2k$ and are included in the sum $\sum \chi^{2\lambda}$. since these two partitions are associates of each other, they restrict to the same irreducible character of A_{2k}. Hence the irreducible constituents in the above decomposition are not distinct.

We remark that various other pairs of groups and subgroups can be investigated by the considering the decomposition of the permutation characters involved. Some of these results will be included in a later paper.

4. BOSE-MESNER ALGEBRAS

Let us consider the matrices in the centralizer algebra of a transitive permutation group G acting on a set of points X. The action of G on X induces an action on the cartesian product $X \times X$ via $(x, y)^g = (x^g, y^g)$ for $x, y \in X$ and $g, \in G$. *Let* $O_0, O_2, \ldots, O_d$ be the orbits of G on $X \times X$, where $O_0 = \{(x,x) | x \in X\}$. With each set O_i, we associate a matrix A_i whose rows and columns are indexed by the elements of X and whose (x, y)-entry is $(A_i)_{x,y} = 1$ if $(x, y) \in O_i$ and $(A_i)_{x,y} = 0$, otherwise. In particular, A_0, is the identity matrix of size $|X|$. We call the A_i the adjacency matrices. These matrices span the centralizer algebra.

One can verify that the matrices in the centralizer algebra C satisfy the following:

1. C contains the identity matrix I and matrix J with all entries equal to 1,
2. C is closed under the Hadamard or entry-wise product,
3. C is closed under ordinary matrix product, which is commutative when restricted to C, and
4. C is closed under transposition.

Using the language of association schemes (Bannai, Ito, 1984), the centralizer algebra above is the Bose-Mesner algebra of a commutative association scheme coming from a transitive permutation group. Association schemes are combinatorial objects that are obtained not only from permutation groups but also arise in the context of algebraic codes, designs and graphs. Each scheme on a finite set X gives rise to a subalgebra of the full matrix algebra $M(X)$ of complex-valued matrices whose entries are indexed by X (the Bose-Mester algebra) that satisfies the above conditions.

Conversely, it is easy to show that any vector subspace C of $M(X)$ satisfying the four properties above is the Bose-Mesner algebra of some associations scheme on X. Thus any such subspace will be called a Bose-Mesner algebra on X. One of the motivations for studying these structures is their recently established connection with spin models from statistical mechanics that yield invariants of knots and linkes (Bannai, Bannai, Jaege, 1997.)

5. CONCLUSION

The centralizer algebra is an interesting algebraic object that is valuable in representation theory and other areas of mathematics. One may study this object from the point of view of Hecke algebras, which are more general structures. Determination of the structure of the algebra, that includes computing its dimension and finding a basis, is an important first step. A bigger goal involves the determination of the character table of commutative Hecke algebras. The process is equivalent to knowing lal the spherical functions on the coset space $H \setminus G$ that could be done in future work. This problem is the finite analogue of the problem of determining the spherical functions of a compact symmetric space.

Abstracting the main properties of the algebras leads to the notion of Bose-Mesner algebras, that historically originated from combinatorial structures called association schemes. Bose-Mesner algebras are also studied today in the construction of spin models from statistical mechanics whose partition functions yield invariants of knots and links.

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