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HENSTOCK INTEGRATION IN A HILBERTIAN COUNTABLY NORMED SPACE WITH NUCLEARITY

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ABSTRACT

Henstock integration of real-valued functions has been extended to functions with values in normed spaces. Cao, who considered Banach-valued functions, showed that Henstock's lemma, which plays an important role in the real-valued case, does not always hold in infinite dimensional Banach spaces. Nakanishi showed that Henstock's lemma holds in a ranked space called Hilbertian CN-space with nuclearity. In this paper, we revisit this space, define r-differentiability of a function with values in an r-separated ranked space, and give results concerning the primitives of Henstock integrable functions with values in this space. Further, we shall give a descriptive definition of the Henstock integral defined by Nakanishi.

Key words: Henstock integral, ranked CN-space, Hilbertian, nuclearity, r-separated, r-differentiable, HL-integral, Strong Lusin

INTRODUCTION

Nakanishi (1994) used a special ranked space called Hilbertian CN-space with nuclearity and defined the Henstock integral of a function with values in the said space. She showed that Henstock's lemma, which does not always hold in infinite dimensional spaces, holds in such a space. In this paper, we give another look at ranked spaces as well as concepts associated with them, and recall some of the important results obtained by Nakanishi. Some of these results will be used to prove our present results concerning the primitives of Henstock integrable functions. As a main result of this study, we characterize the Henstock integral defined by Nakanishi in terms of the Strong Lusin condition of a function.

Preliminaries

Definition 2.1 Let X be a nonempty set such that, for each $x \in X$, there exists a nonempty class $V(x)$ consisting of subsets of X, called ***preneighborhoods*** of x such that $x \in U(x)$ whenever $U(x) \in V(x)$. Put $V = \cup_{x \in X} V(x)$ Suppose further that for each $n \in N$ ($N = \{0,1,....,\}$), there is assigned a nonempty class $V_n \subset V$ satisfying the following: For every $U(x) \in V(x)$ and for every n, there exists a $V(x) \in V_m$ for some $m > n$ such that $V(x) \subset U(x)$. Then the space X endowed with the classes $V(x)$ and V_n for each $x \in X$ and for each $n \in N$ is called a ***ranked space***. It is sometimes denoted by the ordered-triple (X,V,V_n). Further, if U(x) is a preneighhorhood of x and $U(x) \in V_n$, then we say that it is of ***rank*** n. In this case, x is the ***center*** of U(x).

Example 2.2 Let X = [a,b]. For each $x \in X$, let $V(x)$ be the (usual) neighborhood system of x and for each $n \in N$, let $V_n = \{(x - 1/2^{n+1}, x + 1/2^{n+1}) \cap X: x \in [a,b]\}$. If V is the union of all $V(x)$, then (X, V, V_n) is a ranked space.

Definition 2.3 A sequence of preneighborhoods $\{U_i,(x_i,n(i))\}$, i.e., a sequence of preneighborhoods U_i of x_i with ranks n(i), is called a ***fundamental sequence*** (f.s. for brevity) if it satisfies the following conditions:

(C1) The sequence of preneighborhoods is decreasing, i.e., $U_0 \supset U_1 \supset \ldots$;
(C2) $n(0) < n(1) <... < n(k) < n(k+1) <....;$ and
(C3) for every $n \in N$, there exists a k such that $k \geq n$, $x_k = x_{k+1}$ and $n(k) < n(k+1)$.

Definition 2.4 A ranked space (X, V, V_n) is said to be ***r-separated*** if it satisfies the ff. condition: For every $x,y \in X$, $x \neq y$, and for every f.s. $\{U_i(x)\}$ of center x and f.s. $\{V_i(y)\}$ of center y, there exists a k such that $U_k(x) \cap V_k(y) = \emptyset$.

Definition 2.5 Let X be a vector space with a countable sequence of compatible norms $\{p_n\}$. Then X is called a ***countably normed space.*** It is sometimes denoted by $(X,\{p_n\})$. Further, in this space, we have the ff:

(a) A sequence $\{x_j\}$ in X is a ***convergent sequence*** if there is a vector $x \in X$ such that $p_n(x_j - x) \to 0$ as $j \to \infty$ for every norm p_n.
(b) A sequence $\{x_j\}$ in X is a ***Cauchy sequence*** in X if it is a Cauchy sequence for every norm p_n.
(c) X is ***complete*** if every Cauchy sequence in X converges.

Theorem 2.6 [Nakanishi] Let X be a CN-space with a sequence $\{p_n\}$ of increasing norms, i.e., $p_0(x) \leq p_1(x) \leq \ldots$ for every $x \in X$. Then $(X, V(x), V_n)$ is a ranked space, where

$$V(x) = \{x + S_n;\ n \in N\}\ (x \in X),$$
$$V_n = \{x + S_n : x \in X)\ (n \in N),\ \text{and}$$
$$S_n = \{\ y \in X: p_n(y) < 1/2^n\ \}\ (n \in N).$$

Definition 2.7 Let X be a CN-space with a sequence $\{p_n\}$ of increasing sequence of norms, i.e., $p_0(x) \leq p_1(x) \leq \ldots$ for every $x \in X$. We call the ranked space $(X, V(x), V_n)$ described in Theorem 2.6 as a ***ranked countably normed space*** or simply ***ranked CN-space.***

Lemma 2.8 Every ranked CN-space $(X, \{p_n\})$ is r-separated.

Definition 2.9 Let X be an r-separated ranked space, and f: $[a,b] \to X$ a function. We say that the ***r-limit*** of f(t) is L, as t tends to t*, if for every f.s. $\{U_j(t^*)\}$ of center t* in [a,b], there is a f.s. $\{V_j(L)\}$ of center L in X such that $\{f(U_j(t^*)\} < \{V_j(L)\}$. In this case we write,

$$r - \lim_{t \to t^*} (f(t) = L.$$

Definition 2.10 Let X be an r-separated ranked space, and F: $[a,b] \to X$. F is r-differentiable at the point $t^* \in [a,b]$ if

$$r - \lim_{t \to t^*} \frac{F(t)-F(t^*)}{t-t^*} = F_r'(t^*) \text{ exists}$$

F is ***r-differentiable*** on [a,b] if it is r-differentiable at every point in [a,b].

Definition 2.11 Let $(X, \{p_n\})$ be a complete CN-space such that $\{p_n\}$ is an increasing sequence of compatible norms. If each norm p_n is induced by an inner product $(\ ,\ )_n$ on X, then we call $(X, \{p_n\})$ a ***Hilbertian CN-space.***

Let $(X, \{p_n\})$ be a Hilbertian CN-space. Note that the term "Hilbertian" comes from the fact that if X_n is the completion of X with respect to p_n, then X_n is a Hilbert space. Further, from Gel'fand & Shilov's Generalized Functions, if $\{p_n\}$ is an increasing sequence of compatible norms, then the sequence $\{X_n\}$ can be considered to have the relationship $X_0 \supset X_1 \supset \ldots \supset X$. Hence, for $m < n$, the mapping $\varphi_{m,n}: X \to X$ defined by $\varphi_{m,n}(x_{(n)}) = x_{(m)}$, where $x_{(n)}$ and $x_{(m)}$ denote the same element x of X considered as element of X_n and X_m respectively, is a continuous linear operator from the everywhere dense subset X of X_n, onto the everywhere dense subset X of X_m. Thus, by the Hahn Banach Theorem, $\varphi_{m,n}$ can be extended to a continuous linear operator $T_{m,n}$ from X_n onto a dense subset of X_m

Definition 2.12 Let $(X,\{p_n\})$ be a Hilbertian CN-space with increasing sequence $\{p_n\}$ of compatible norms. We say that $(X,\{p_n\})$ has the ***nuclearity*** property if for each m, there exists an $n > m$ satisfying the following property:

(N) $T_{m,n}(x) = \sum_{k=1}^{\infty} \lambda_{m,n,k} (x, e_{n,k})_n e_{m,k}$ for every $x \in X$, where $(,)_n$ is the inner product on X_n that induces p_n, $\{e_{n,k}\}$ and $\{e_{m,k}\}$ are orthonormal systems of vectors in the spaces X_n and X_m, respectively, each $\lambda_{m,n,k} > 0$, and the series $\sum_{k=1}^{\infty} \lambda_{m,n,k}$ converges.

Definition 2.13 Let $(X,\{p_n\})$ be a complete ranked CN-space. An X-valued function f defined on [a,b] is said to be ***Henstock integrable*** to a vector $z \in X$ on [a,b] if for every $n \in N$ there exists $\delta_n(\xi) > 0$ on [a,b] such that for any δ-fine division $D = \{([u,v];\xi)\}$ of [a,b], we have

$$p_n((D)\sum f(\xi)(v - u) - z) < 1/2^n.$$

Definition 2.14 Let (X,p) be a normed space. An X-valued function f defined on [a,b] is said to be strongly ***Henstock integrable*** on [a,b] if there exists an additive function $F:[a,b] \rightarrow X$ satisfying the following property: For every $\varepsilon > 0$ there exists $\delta(\xi) > 0$ on [a,b] such that for any δ-fine division $D = \{([u,v];\xi)\}$, we have

$$(D)\sum p(f(\xi)(v - u) - F(v) + F(u)) < \varepsilon.$$

The above integral is known as the ***HL-integral*** and the function F, where we assume that $F(a) = 0$, is called the ***HL-primitive*** of f. The following theorem is due to Cao.

Theorem 2.15 [Cao] Let (X,p) be a Banach space. If $f: [a,b] \rightarrow X$ is strongly Henstock integrable with HL-primitive F on [a,b], then F is differentiable almost everywhere on [a,b] and $F'(t) = f(t)$ a.e. on [a,b].

Definition 2.16 A function $F: [a,b] \rightarrow X$ is said to satisfy the ***Strong* Lusin** condition if for every subset E of [a,b] of measure zero and for every $\varepsilon > 0$, there exists a $\delta(\xi) > 0$ such that for any δ-fine division $D = \{([u,v];\xi)\}$ of [a,b] with $\xi \in E$, we have

$$(D)\sum p(F(v) - F(u)) < \varepsilon.$$

Theorem 2.17 [Canoy] If $F: [a,b] \rightarrow X$ is a primitive of a strongly Henstock integrable function f, then F satisfies the Strong Lusin condition and $F'(x) = f(x)$ a.e. on [a,b].

Theorem 2.18 [**Canoy**] A function $f : [a,b] \to X$ is strongly Henstock integrable on $[a,b]$ if and only if there is an SL-function $F: [a,b] \to X$ such that $F'(x) = f(x)$ a.e. on $[a,b]$.

3. RESULTS

The first three results are proved by the author in his earlier paper.

Lemma 3.1 Let $(X, \{p_n\})$ be a ranked CN-space. If an X-valued function F defined on $[a,b]$ is differentiable at a point $t^* \in [a,b]$ with respect to p_n, then it is differentiable there with respect to p_m for all $m < n$. Moreover, $F_m'(t^*) = F_n'(t^*)$ $(m < n)$, where $F_m'(t^*)$ denotes the derivative of F at t^* with respect to p_m.

A direct consequence of the above lemma is the following

Corollary 3.2 Let $(X, \{p_n\})$ be a ranked CN-space. If an X-valued function F defined on $[a,b]$ is differentiable at a point $t^* \in [a,b]$ for every p_n, then $F_n'(t^*) = F_m'(t^*)$ for all m and n.

Theorem 3.3 Let $(X, \{p_n\})$ be a ranked CN-space. An X-valued function F defined on $[a,b]$ is r-differentiable at $t^* \in [a,b]$ if and only if it is differentiable at t^* for every p_n. Moreover, $F_n'(t^*) = F'(t^*)$ for every n, where $F_n'(t^*)$ denotes the derivative of F at t^* with respect to p_n.

The following theorem is due to Nakanishi.

Theorem 3.4 (Henstock's Lemma) Let $(X, \{p_n\})$ be a Hilbertian CN-space with nuclearity, and $f: [a,b] \to X$ a Henstock integrable function on $[a,b]$ with primitive $F(t) = \int_a^t f(s)ds$. Then, for every n, there exists a positive function $\delta_n(\xi)$ on $[a,b]$ such that for any δ_n-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$(D)\sum p_n(f(\xi)(v-u) - F(v) + F(u)) < 1/2^n.$$

Corollary 3.5 Let $(X, \{p_n\})$ be a Hilbertian CN-space with nuclearity, and $f: [a,b] \to X$ a Henstock integrable function on $[a,b]$ with primitive $F(t) = \int_a^t f(s)ds$. Then for every n, the following holds: Given $\varepsilon > 0$, there exists a positive function $\delta(\xi)$ on $[a,b]$ such that for any δ-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$(D)\sum p_n((f(\xi)(v-u) - F(v) + F(u)) < \varepsilon.$$

Lemma 3.6 Let $(X, \{p_n\})$ be CN-space with $p_1(x) < p_2(x) < \ldots$ for all x in X, and $f: [a,b] \to X$. If f is strongly Henstock integrable on $[a,b]$ with HL-primitive F_n with respect to p_n, then it is strongly Henstock integrable on $[a,b]$ with respect to p_m for every $m < n$. Moreover, $F_m = F_n$ on $[a,b]$ for all $m < n$.

Theorem 3.7 Let $(X, \{p_n\})$ be a Hilbertian CN-space with nuclearity, and $f: [a,b] \to X$. Then f is Henstock integrable on [a,b] with primitive F if and only if f is strongly Henstock integrable on [a,b] with HL-primitive F for every p_n.

Proof: ($\Rightarrow$) Suppose f is Henstock integrable on [a,b] with primitive

$F(t) = \int_a^t f(s)ds$. Then, by Corollary 3.5, f is strongly Henstock integrable on [a,b] with HL-primitive F for every norm p_n.

($\Leftarrow$) Suppose that f is strongly Henstock integrable on [a,b] for every p_n. By Lemma 3.6, we may take a common HL-primitive $F = F_1$. Then for each n, there exists a $\delta_n(x) > 0$ such that for any δ_n-fine division $D = \{([u,v];\xi)\}$ of [a,b], we have

$$(D)\sum p_n((f(\xi)(v-u) - F(v) + F(u)) < 1/2^n.$$

Thus,

$$\begin{aligned} p_n((D)\sum f(\xi)(v-u) - F(b) + F(a)) &= p_n((D)\sum(f(\xi)(v-u) - F(v) + F(u))) \\ &\le (D)\sum(p_n(f(\xi)(v-u) - F(v) + F(u)) \\ &< 1/2^n \end{aligned}$$

for all n. Therefore, f is Henstock integrable to F(b) – F(a) = F(b) on [a,b]. □

Theorem 3.8 Let $(X,\{p_n\})$ be a Hilbertian ranked CN-space with nuclearity, and $f: [a,b] \to X$ a Henstock integrable function on [a,b] with primitive $F(t) = \int_a^t f(s)ds$. Then F is r-differentiable almost everywhere on [a,b].

Proof: By Theorem 3.7, f is strongly Henstock integrable on [a,b] with HL-primitive F for every p_n. Thus, by Theorem 2.15, F is differentiable almost everywhere on [a,b] for every p_n. By Theorem 3.3, F is r-differentiable almost everywhere on [a,b]. □

Theorem 3.9 Let $(X,\{p_n\})$ be a Hilbertian CN-space with nuclearity and $f : [a,b] \to X$ a function. Then f is Henstock integrable on [a,b] if and only if there exists an SL-function $F: [a,b] \to X$ such that $F_r'(x) = f(x)$ a.e. on [a,b].

Proof: ($\Rightarrow$) Suppose f is Henstock integrable with primitive F. By Theorem 3.7, f is strongly Henstock integrable with primitive F for every norm p_n. Now, by Theorem 2.17, F is an SL-function and $F'(t) = f(t)$ a.e. on [a,b] for every norm p_n. By Theorem 3.3, $F_r'(t) = f(t)$ a.e. on [a,b].

($\Leftarrow$) Suppose there exists an SL function F such that $F_r'(t) = f(t)$ a.e. on [a,b]. By Theorem 3.3, $F'(t) = f(t)$ a.e. on [a,b] for every norm p_n. It follows from Theorem 2.18 that f is strongly Henstock integrable on [a,b] with HL-primitive F for each norm p_n. Thus, by Theorem 3.7, f is Henstock integrable on [a,b]. □

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